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What is the Smale Paradox, Why is it Important, and How can it be Resolved

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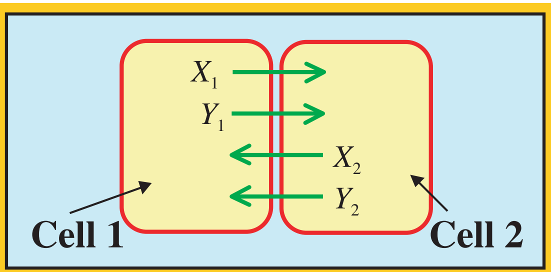
Turing's Two-Cell Reaction-Diffusion Equations

$\boxed{\begin{matrix} X \\ Y \end{matrix}}$ a second-order reaction cell

Turing's equations:
$$\frac{d}{dt}X = \boxed{5X - 6Y} + I_1$$
$$\frac{d}{dt}Y = \boxed{6X - 7Y} + I_2$$

the uncoupled reaction cell has a unique globally asymptotically stable operating point Q for any choice for the DC terms I_1 and I_2



Cell 1

$$\dot{X}_1 = (5X_1 - 6Y_1 + 1) + 0.5(X_2 - X_1)$$
$$\dot{Y}_1 = (6X_1 - 7Y_1 + 1) + 4.5(Y_2 - Y_1)$$

Cell 2

$$\dot{X}_2 = (5X_2 - 6Y_2 + 1) + 0.5(X_1 - X_2)$$
$$\dot{Y}_2 = (6X_2 - 7Y_2 + 1) + 4.5(Y_1 - Y_2)$$

$$\frac{d}{dt}\begin{bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \end{bmatrix} = \boldsymbol{G_1}\begin{bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \end{bmatrix} + \boldsymbol{G_2}\begin{bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \end{bmatrix} = \boldsymbol{G}\begin{bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \end{bmatrix}$$

$$\boldsymbol{G_1} = \begin{pmatrix} 5 & -6 & 0 & 0 \\ 6 & -7 & 0 & 0 \\ 0 & 0 & 5 & -6 \\ 0 & 0 & 6 & -7 \end{pmatrix} \text{ stable} \qquad \boldsymbol{G_2} = \begin{pmatrix} -0.5 & 0 & 0.5 & 0 \\ 0 & -4.5 & 0 & 4.5 \\ 0.5 & 0 & -0.5 & 0 \\ 0 & 4.5 & 0 & -4.5 \end{pmatrix} \text{ stable}$$

$$\boldsymbol{G} \triangleq \boldsymbol{G_1} + \boldsymbol{G_2} = \begin{pmatrix} 4.5 & -6 & 0.5 & 0 \\ 6 & -11.5 & 0 & 4.5 \\ 0.5 & 0 & 4.5 & -6 \\ 0 & 4.5 & 6 & -11.5 \end{pmatrix} \text{ unstable}$$

What are the mechanisms behind these diffusion-driven instabilities?

Hint: the key factor is the choice of right coefficients in the reaction cell

A.M. Turing, "The chemical basis of morphogenesis," Phil. Trans. Roy. Soc. B, vol. 237, pp. 37–72, Aug. 1952

What is the Smale Paradox

Object of study: two biological cells immersed in a dissipative medium

Model: a nonlinear form of Turing's equations:

$$\begin{aligned}\frac{d}{dt}\boldsymbol{z_1} &= \boxed{\boldsymbol{R}(\boldsymbol{z_1})} + \boxed{\boldsymbol{\mu}\,(\boldsymbol{z_2}-\boldsymbol{z_1})}\\ \frac{d}{dt}\boldsymbol{z_2} &= \boxed{\boldsymbol{R}(\boldsymbol{z_2})} + \boxed{\boldsymbol{\mu}\,(\boldsymbol{z_1}-\boldsymbol{z_2})}\end{aligned}, \quad \boldsymbol{z_1},\boldsymbol{z_2} \in \mathbb{R}^4 : \boldsymbol{z_1},\boldsymbol{z_2} \geq \boldsymbol{0}\ , \quad \boldsymbol{\mu} = \begin{pmatrix}\mu_1 & 0 & 0 & 0\\ 0 & \mu_2 & 0 & 0\\ 0 & 0 & \mu_3 & 0\\ 0 & 0 & 0 & \mu_4\end{pmatrix}, \quad \mu_i > 0\ , \quad i \in \{1,2,3,4\}$$

reaction diffusion

- The kinetic equation of each reaction cell is described by a nonlinear vector field in $\mathbb{R}^4$.
- For any initial condition the uncoupled cell approaches a globally asymptotically stable (GAS) equilibrium Q
- Adjoining the two cells along a membrane, as long as $\boldsymbol{z_1}(0) \neq \boldsymbol{z_2}(0)$, the two-cell array features a certain oscillatory behaviour for a given choice of the diffusion coefficient matrix $\boldsymbol{\mu}$.

S. Smale, 1974: "There is a paradoxical aspect to the example. One has two dead (mathematically dead) cells interacting by a diffusion process, which has a tendency in itself to equalize the concentrations. Yet in interaction, a state continues to pulse indefinitely."

S. Smale, "A Mathematical Model of Two Cells via Turing's Equation," American Mathematical Society, Lectures in Applied Mathematics, vol. 6, pp. 15-26, 1974

The Smale Paradox: A 50-Year Old Problem

S. Smale, 1974: "Various forms of Turing's equations, or reaction-diffusion equations have appeared in one form or another in many works and fields."

"However, any sort of systematic understanding or analysis seems far away."

"Before one can expect any general understanding, many examples will have to be thought through, both on the mathematical side and on the experimental side."

"The work here poses a sharp problem, namely to axiomatize the properties necessary to bring about oscillation via diffusion"

S. Smale, "A Mathematical Model of Two Cells via Turing's Equation," American Mathematical Society, Lectures in Applied Mathematics, vol. 6, pp. 15-26, 1974

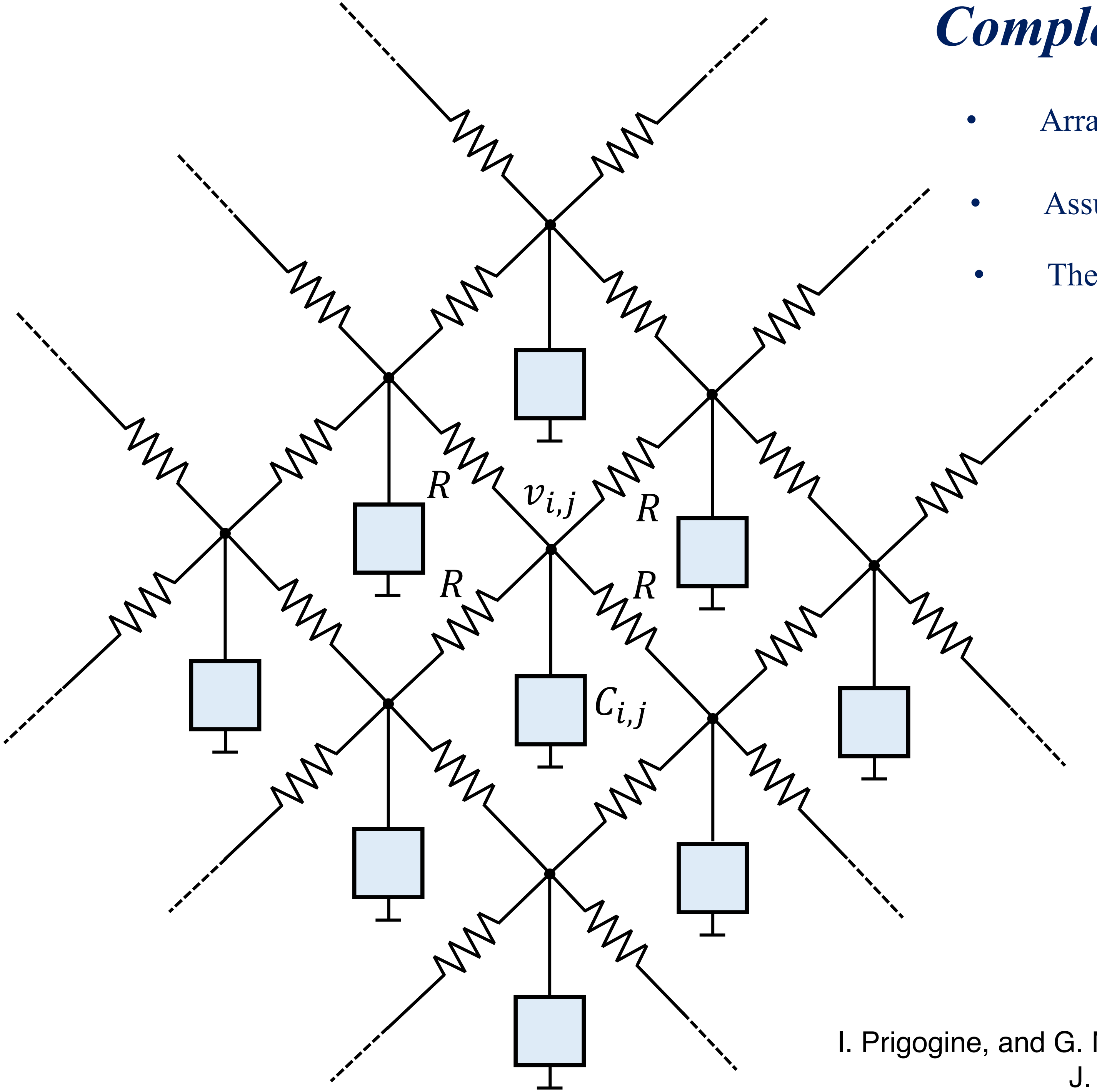
Ascoli, Tetzlaff, Demirkol, Chua, 2022:

Answer: it is necessary that the reaction cell is locally-active at the globally asymptotically stable operating point Q

The cell is then said to be poised on Edge of Chaos at the operating point Q

A. Ascoli, A.S. Demirkol, R. Tetzlaff, and L.O. Chua, "Edge of Chaos Resolves Smale Paradox," IEEE Trans. on Circuits and Systems-I: Regular Papers, 2022 DOI: 10.1109/TCSI.2021.3133627

Complexity in Reaction-Diffusion Systems



- Array of diffusively-coupled regularly-spaced identical reaction cells
- Assume $C_{i,j}$ to feature a GAS operating point, with $\lim_{t\to\infty} v_{i,j} = V$
- The array would then admits the homogeneous solution, i.e.

$$\lim_{t\to\infty} v_{i,j} = V \quad \forall i \in \{1,\dots,M\}, \forall j \in \{1,\dots,N\}$$

What are the necessary conditions

for the

destabilization

of the

homogeneous solution?

Symmetry-breaking phenomena with spatio-temporal pattern formation in homogeneous media is an example par excellence of complexity

I. Prigogine, and G. Nicolis, “On Symmetry-Breaking Instabilities in Dissipative Systems,” J. Chem. Phys., vol. 46, no. 9, pp. 3542–3550, 1967

Schrödinger, Prigogine, Eigen,
Gell-Mann, Turing, and Smale
have all been searching for a
missing new Physics Principle
to explain Complexity
in physical systems

The Local Activity

and its Pearl,

the Edge of Chaos,

is in fact the

Missing New Principle

Definition of Local Activity

A system is said to be

locally active

if and only if

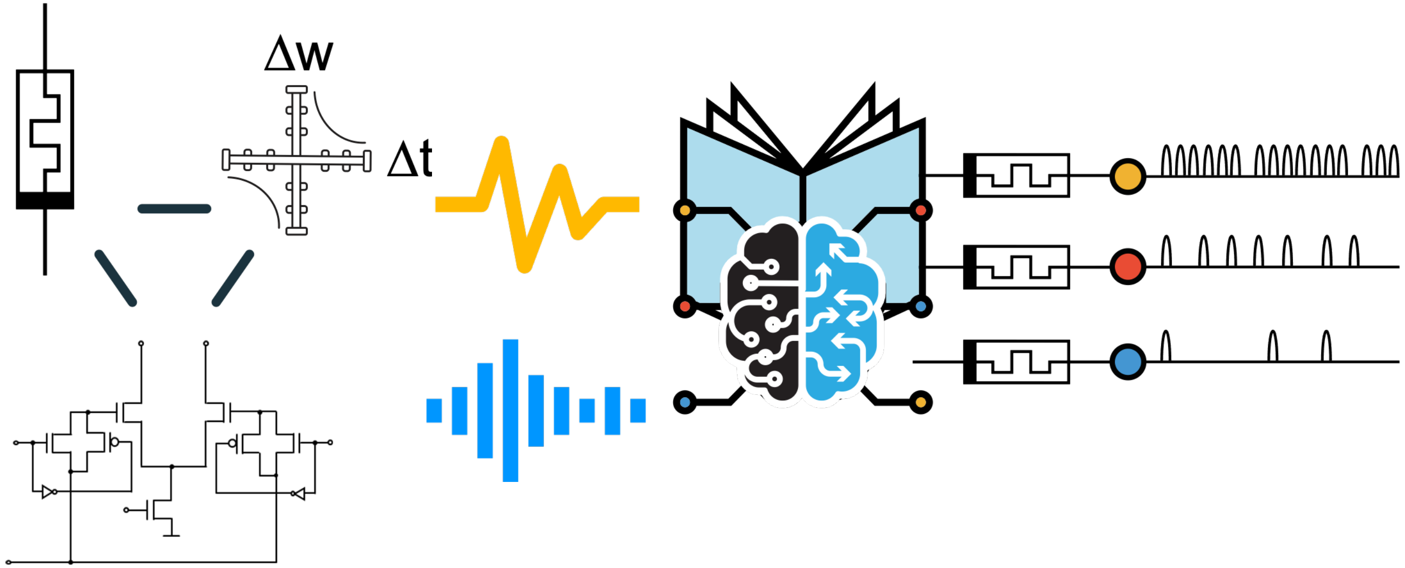
it is capable to

amplify

infinitesimal fluctuations in energy

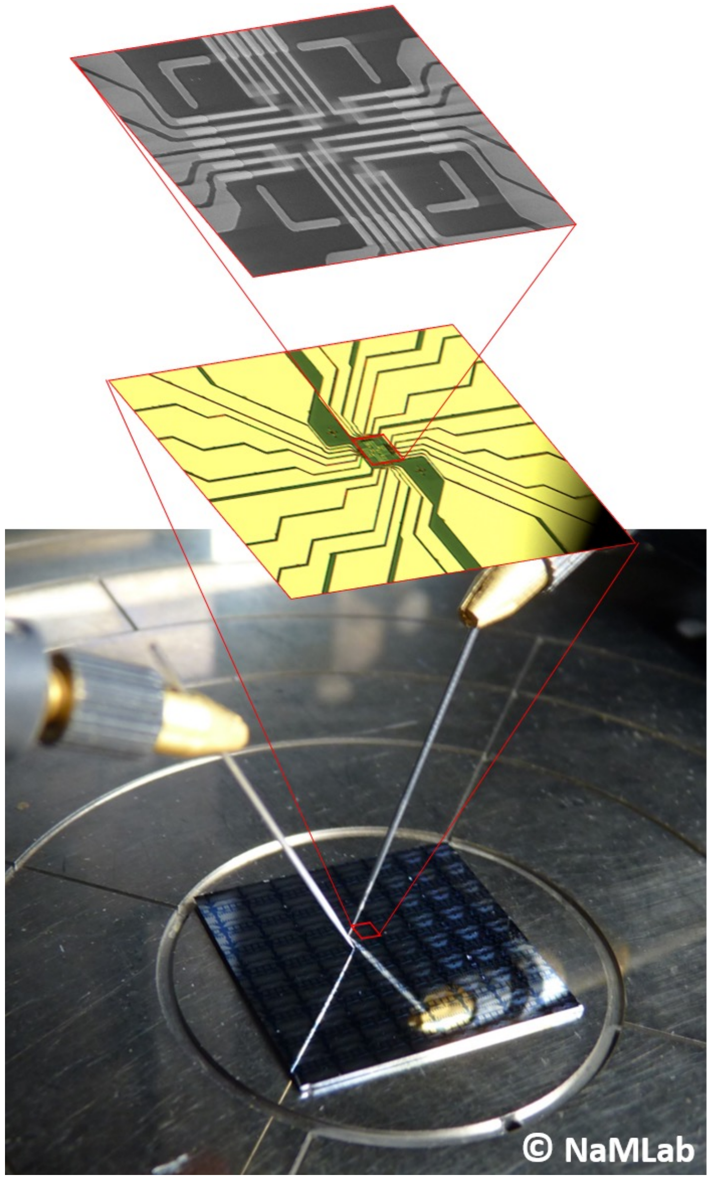
Locally-Active Memristors

- *Particular memristor physical realizations may act as local energy sources*



Locally-active memristors may enable the hardware implementation of bio-inspired computing paradigms (Dr. E. Covi, ERC Starting Grant 2021, MEMRINESS)

- *Let us then apply the local activity definition to a memristor*



Memristors fabricated at NaMLab.
Contact is made to the terminals of one of them for electrical measurements.
The lower (upper) inset shows a microscope image (a scanning electron microscope image) with a zoom-in view of the memristor structures.

Rigorous definition of local activity for a memristor

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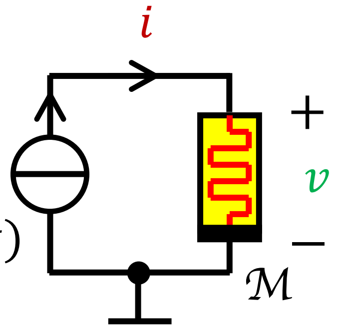
$$\boxed{\begin{aligned} v &= R(\boldsymbol{x}, i) \cdot i \\ \lim_{i \to 0} R(\boldsymbol{x}, i) &\neq \infty \\ \dot{\boldsymbol{x}} &= f(\boldsymbol{x}, i) \end{aligned}}$$

extended memristor

- Let the current-controlled memristor $\mathcal{M}$ be biased at $Q = (V_Q, I_Q)$
- Let a small-signal current $\delta i(t)$ add up to the bias level I_Q



$I_S = I_Q$

$i_S = I_Q + \delta i(t)$

$$i_S(t) \equiv i(t) = I_Q + \delta i(t) \longrightarrow \begin{cases} x(t) = X_Q + \delta x(t) \\ v(t) = V_Q + \delta v(t) \end{cases}$$

memristor overall response

Definition:

The current-controlled memristor $\mathcal{M}$ is locally active at the operating point $Q = (V_Q, I_Q)$ if there exists at least one possible small-signal current $\delta i(t)$, which, superimposed upon the bias level I_Q from $t = t_0$, leads to a <u>negative net energy</u> entering the one-port over $t \in [t_0, t]$, i.e.

$$\delta \mathcal{E}(t_0, \bar{t}) = \int_{t_0}^{\bar{t}} \delta v(\tau) \cdot \delta i(\tau)\, d\tau < 0 \quad ,$$

for some finite time $t = \bar{t}$

- impractical for testing purposes
- fortunately, there exists a powerful theorem, which simplifies this investigation
- for the application of the theorem it is necessary to derive the device small-signal impedance about Q

System-theoretic method to derive the DC voltage-current characteristic of a memristor

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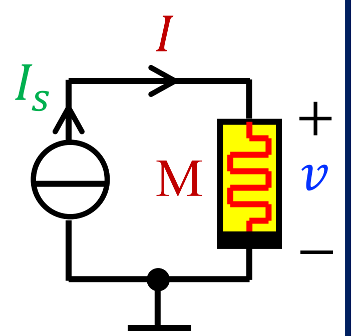
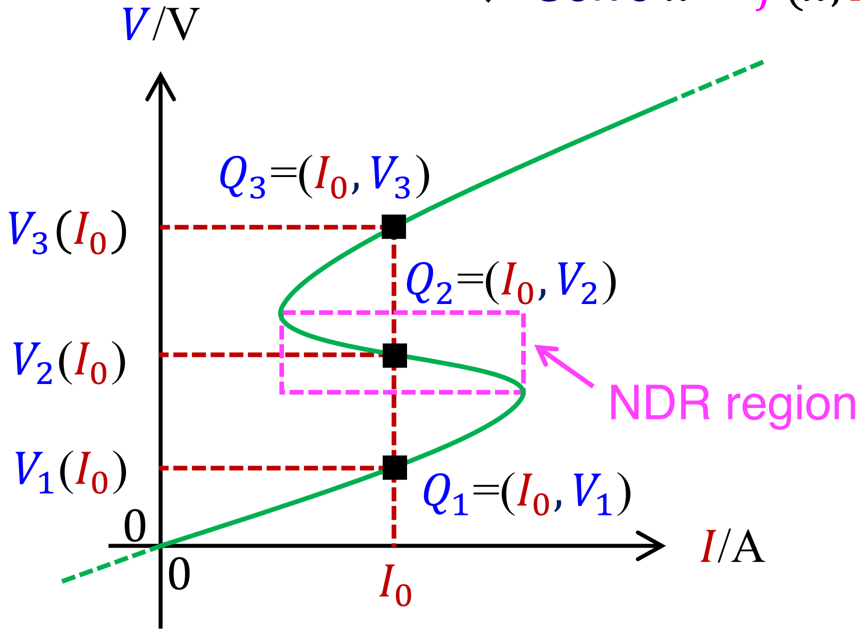
- First-order current-controlled generic memristor model: $\begin{cases} \dot{x} = f(x, i) \\ v = R(x) \cdot i \end{cases}$

- Insert a DC current I_s into the memristor $\Longrightarrow$ Let $I_s \equiv I = I_0$, where I_0 is any real number



$\Longrightarrow$ Solve $\dot{x} = f(x, I_0) = 0$ $\Longrightarrow$ State solutions: $X_1(I_0), \ldots, X_n(I_0)$



DC V-I characteristic of the memristor
(only a positive sweep in I_0 was carried out)

> Note: A negative differential resistance (NDR) region in the DC V-I locus of a one-port is a signature for local activity

$\Downarrow$

Calculate the corresponding voltage values from Ohm's law

$$V_1(I_0) = R\big(X_1(I_0)\big) \cdot I_0, \quad \ldots, \quad V_n(I_0) = R\big(X_n(I_0)\big) \cdot I_0$$

$\Downarrow$

Mark the following points on the current-voltage plane

$$(I_0, V_1(I_0)), \quad \ldots, \quad (I_0, V_n(I_0))$$

$\Downarrow$

Repeat the above procedure for each value of $I_0 \in (-\infty, \infty)$

$\Downarrow$

Interpolate the current-voltage pairs derived in all the iterations

Small-signal model of a first-order memristor

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$$\begin{cases} \dot{x} = f(x,i),\ x \in \mathbb{R} \\ \lim\limits_{i \to 0} R(x,i) \neq \infty \\ v = v(x,i) = R(x,i) \cdot i \end{cases}$$

DAE set of a first-order extended memristor

- Linearize the DAE set about an operating point $Q = (I_Q, V_Q)$

$$\begin{cases} \dfrac{d\delta x}{dt} = a(Q) \cdot \delta x + b(Q) \cdot \delta i \\ \delta v = c(Q) \cdot \delta x + d(Q) \cdot \delta i \end{cases} \quad \text{where} \quad \begin{cases} a(Q) \triangleq \left.\dfrac{\partial f(x,i)}{\partial x}\right|_Q & b(Q) \triangleq \left.\dfrac{\partial f(x,i)}{\partial i}\right|_Q \\ c(Q) \triangleq \left.\dfrac{\partial v(x,i)}{\partial x}\right|_Q & d(Q) \triangleq \left.\dfrac{\partial v(x,i)}{\partial i}\right|_Q \end{cases}$$

- Transform the linearized system in the Laplace domain (with $\delta x(0) = 0$)

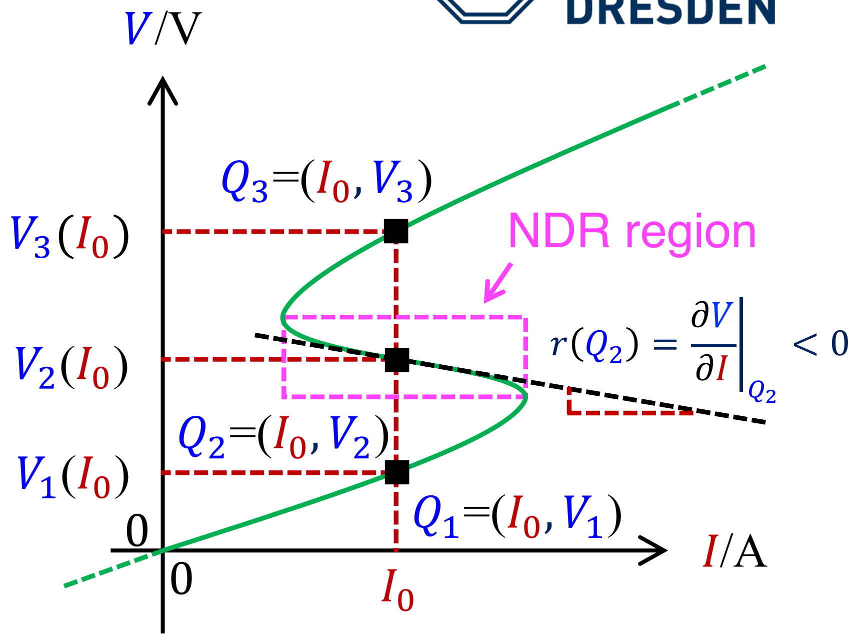
$$s \cdot \mathcal{L}\{\delta x(t)\} = a(Q) \cdot \mathcal{L}\{\delta x(t)\} + b(Q) \cdot \mathcal{L}\{\delta i(t)\} \longrightarrow \mathcal{L}\{\delta x(t)\} = \frac{b(Q)}{s - a(Q)} \cdot \mathcal{L}\{\delta i(t)\}$$

$$\mathcal{L}\{\delta v(t)\} = c(Q) \cdot \mathcal{L}\{\delta x(t)\} + d(Q) \cdot \mathcal{L}\{\delta i(t)\} \longrightarrow \mathcal{L}\{\delta v(t)\} = c(Q) \cdot \frac{b(Q)}{s - a(Q)} \cdot \mathcal{L}\{\delta i(t)\} + d(Q) \cdot \mathcal{L}\{\delta i(t)\}$$

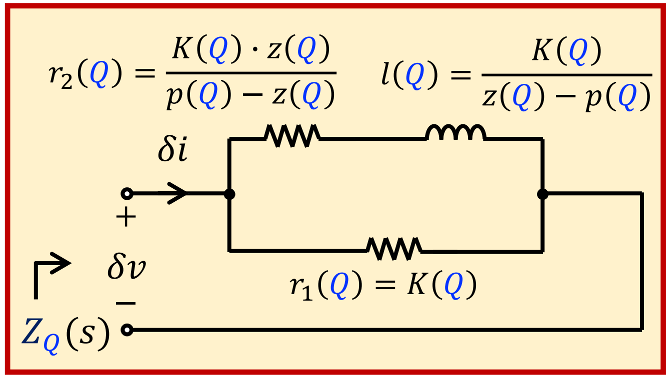
- The local impedance Z_Q of the memristor about Q is computed via

$$\boxed{Z_Q(s) \triangleq \frac{\mathcal{L}\{\delta v(t)\}}{\mathcal{L}\{\delta i(t)\}} = K(Q) \cdot \frac{s - z(Q)}{s - p(Q)}}$$

where $K(Q) = d(Q)$ $\quad z(Q) = \dfrac{a(Q) \cdot d(Q) - b(Q) \cdot c(Q)}{d(Q)}$ $\quad p(Q) = a(Q)$



DC V-I characteristic of the memristor
(only a positive sweep in I_0 was carried out)



Memristor small-signal equivalent circuit model at Q

Small-signal resistance at Q: $\boxed{r(Q) = r_1(Q) \parallel r_2(Q)}$

Local Activity Theorem

A current-driven one-port is Locally Active at Q $\Leftrightarrow$ any one of 4 conditions applies:

1. $Z_Q(s)$ has a pole p such that $\Re\{p\} > 0$

2. $Z_Q(s)$ has a simple pole of the form $p = j\omega_P$, and

$$\mathrm{res}\big(Z_Q(s), p\big) = \lim_{s \to j\omega_P} (s - j\omega_P) \cdot Z_Q(s)$$

is either a complex number or a negative real number

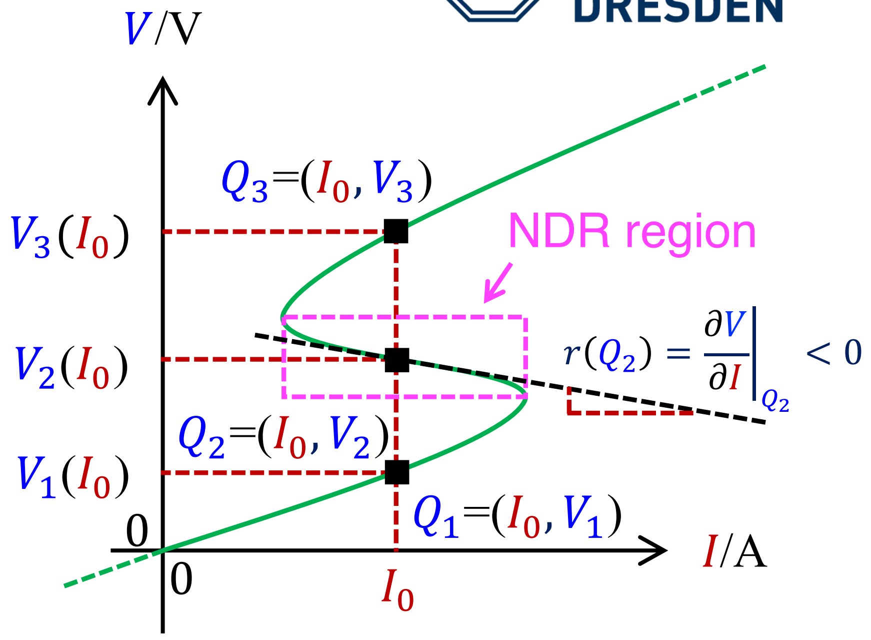
3. $Z_Q(s)$ has a multiple pole of the form $p = j\omega_P$

4. $\Re\{Z_Q(j\omega)\}$ is negative for at least one real-valued angular frequency ω

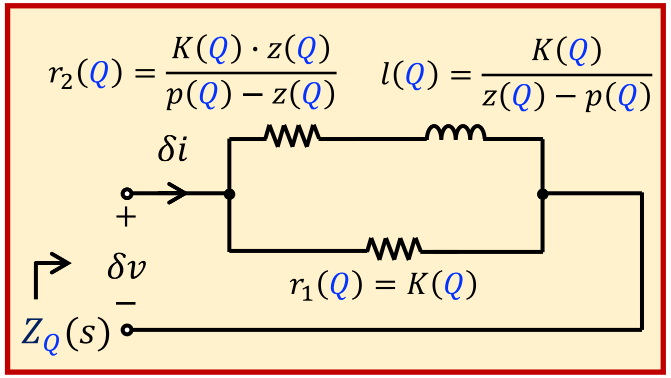
Notes:

- conditions 2. and 3. refer to marginal cases
- condition 4. holds true at an operating point along the NDR region, where

$$\Re\{Z_Q(0)\} \equiv r(Q) < 0$$



DC V-I characteristic of the NaMlab memristor (only a positive sweep in I_0 was carried out)



Memristor small-signal equivalent circuit model at Q

Small-signal resistance at Q: $\boxed{r(Q) = r_1(Q) \parallel r_2(Q)}$

Definition: Edge of Chaos (EOC)

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A one-port

is said to be on the

Edge of Chaos

if

it is locally active at some

asymptotically-stable

operating point Q

(i.e. only condition 4. from the Local Activity Theorem applies)

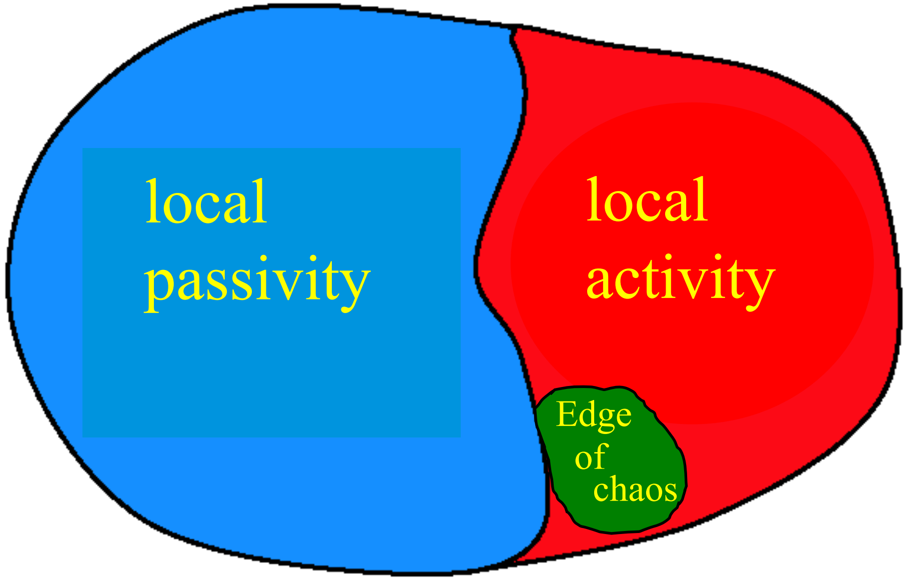
A current-driven one-port is Locally Active at Q $\Leftrightarrow$ any one of 4 conditions applies:

1. $Z_Q(s)$ has a pole p such that $\Re\{p\} > 0$
2. $Z_Q(s)$ has a simple pole of the form $p = j\omega_P$, and

$$\mathrm{res}\big(Z_Q(s), p\big) = \lim_{s \to j\omega_P} (s - j\omega_P) \cdot Z_Q(s)$$

   is either a complex number or a negative real number

3. $Z_Q(s)$ has a multiple pole of the form $p = j\omega_P$
4. $\Re\{Z_Q(j\omega)\}$ is negative for at least one real-valued angular frequency



Edge of Chaos

is the

"Pearl"

embedded within

the domain of

Local Activity

Edge of Chaos

is an

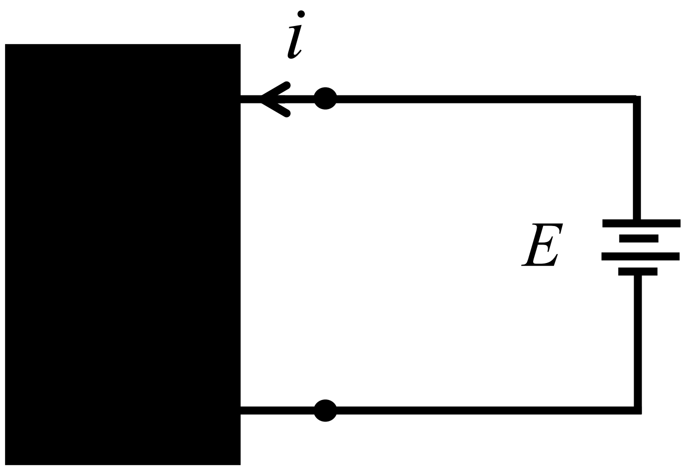
innate characteristic

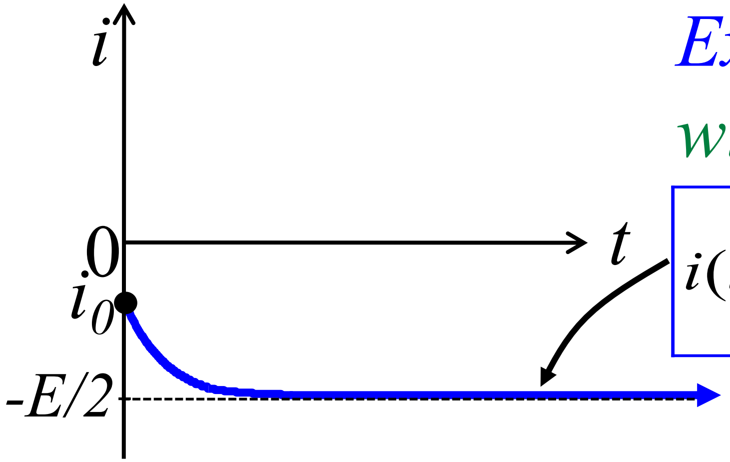
of a

dynamical system

Chua's Riddle

Chua's Riddle

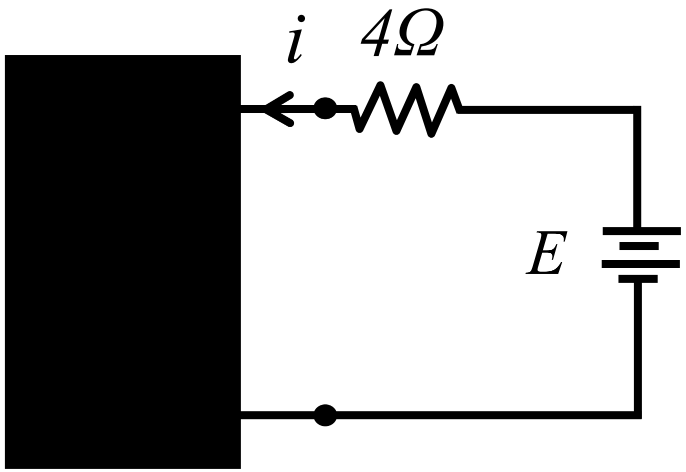


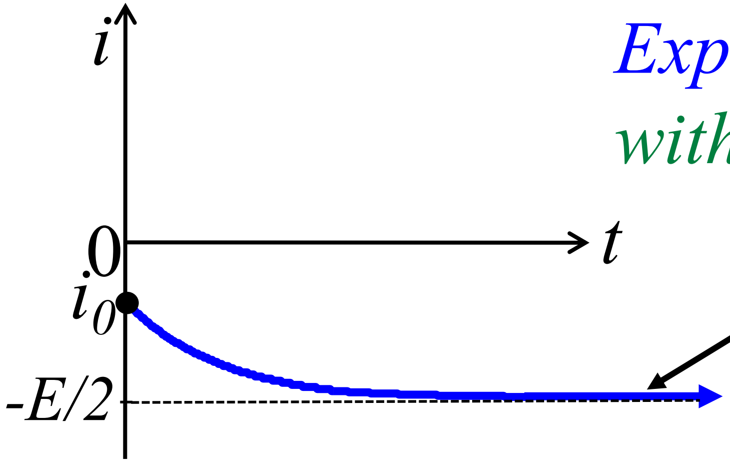


Exponential waveform

with time constant $\tau = 1$

$$i(t) = -\frac{E}{2} + \left[i_0 + \frac{E}{2} \right] e^{-t}$$

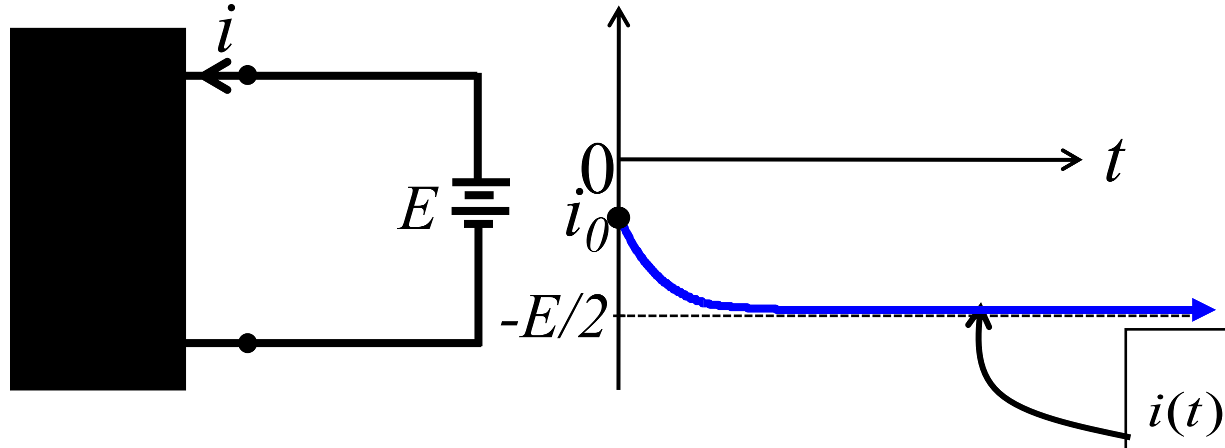




Exponential waveform

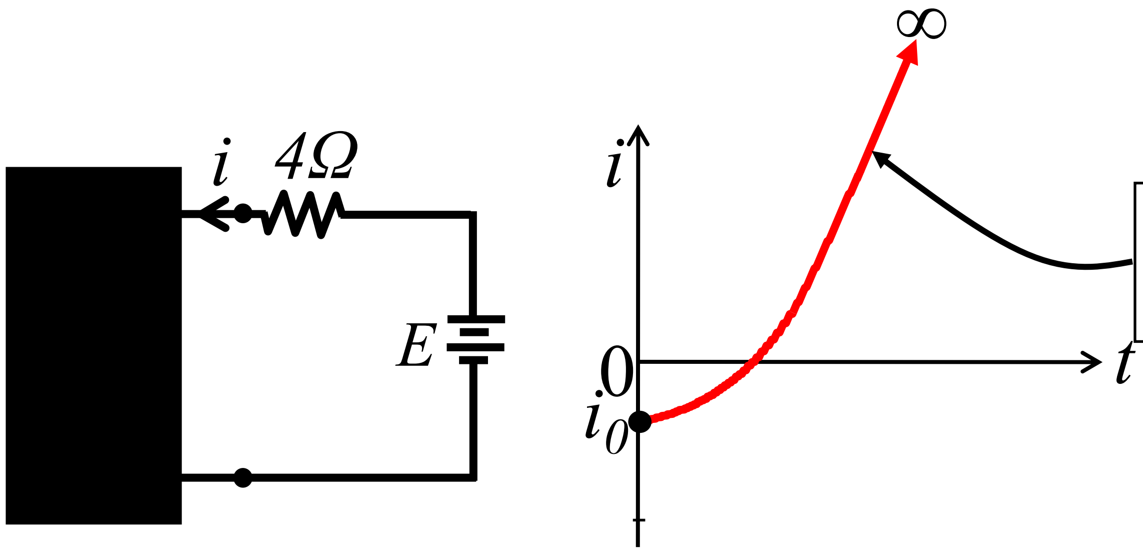
with time constant $\tau > 1$

Chua's Riddle



Exponential waveform with time constant $\tau = 1$

$$i(t) = -\frac{E}{2} + \left[i_0 + \frac{E}{2} \right] e^{-t}$$



$$i(t) = \frac{E}{2} + \left[i_0 - \frac{E}{2} \right] e^{t}$$

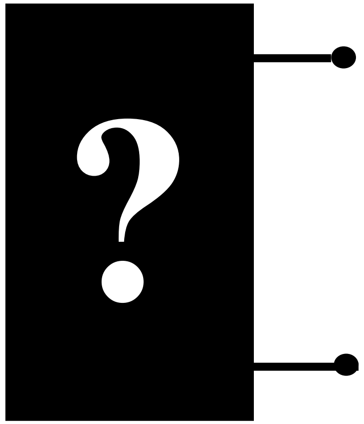
- ***This is yet another example of Complexity. How may this happen?***

What is

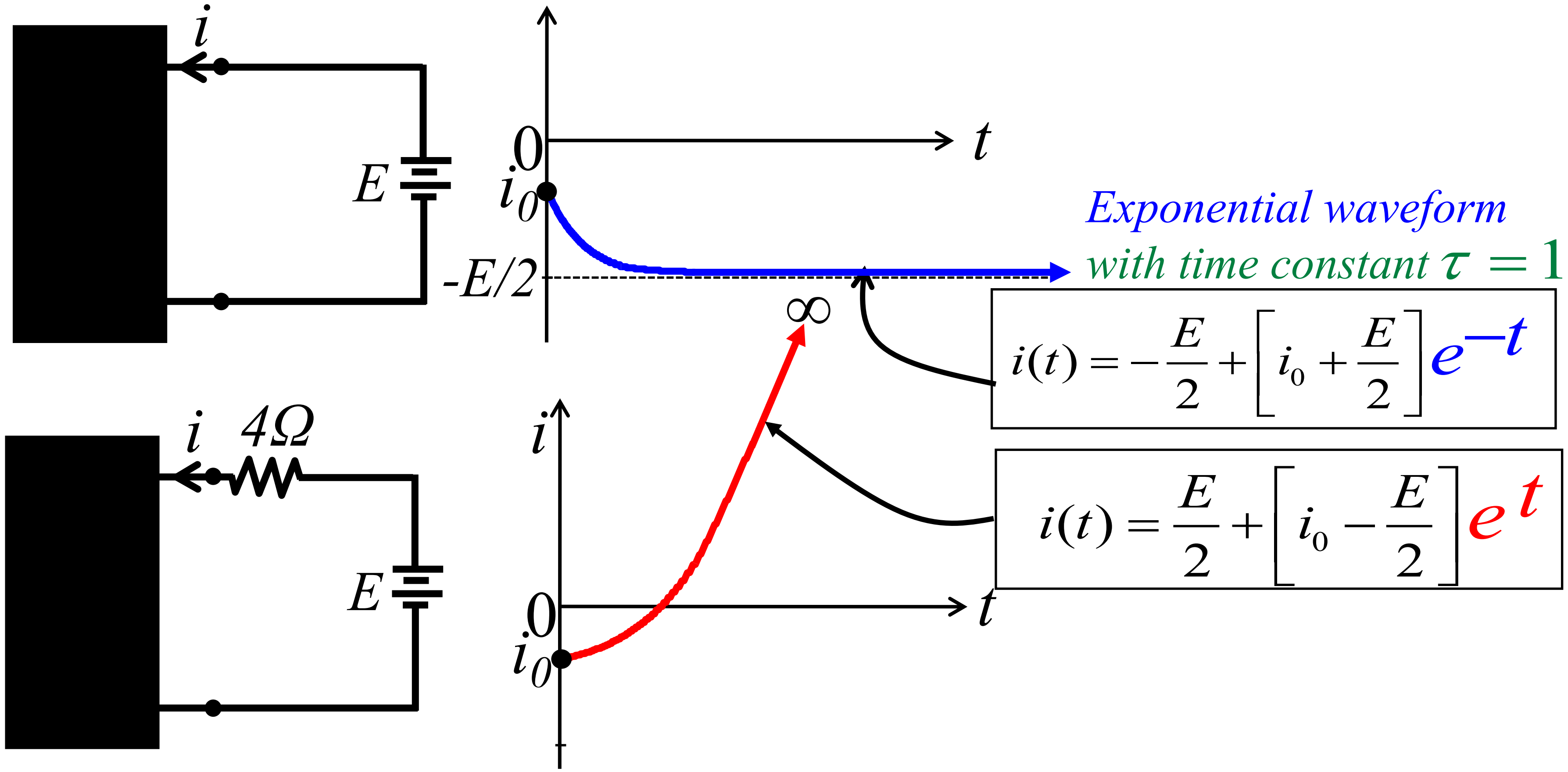
inside

the

Black Box



Chua's Riddle



Hint : the Black Box contains just two basic linear two-terminal circuit elements

The *Answer*

to

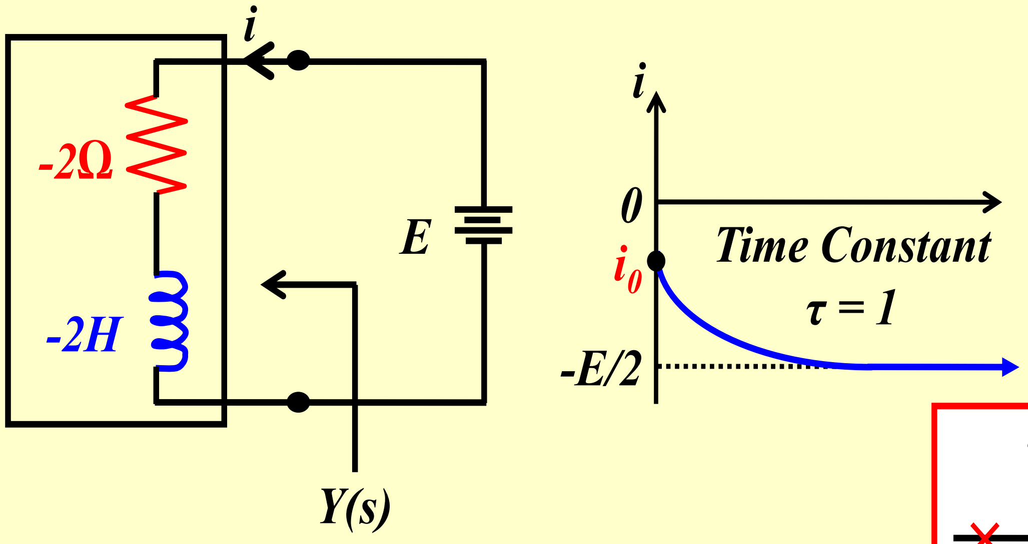
Chua's Riddle

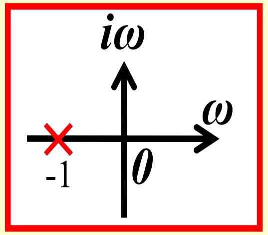
is the Essence of the

Edge of Chaos

Chua's Riddle:

Solution

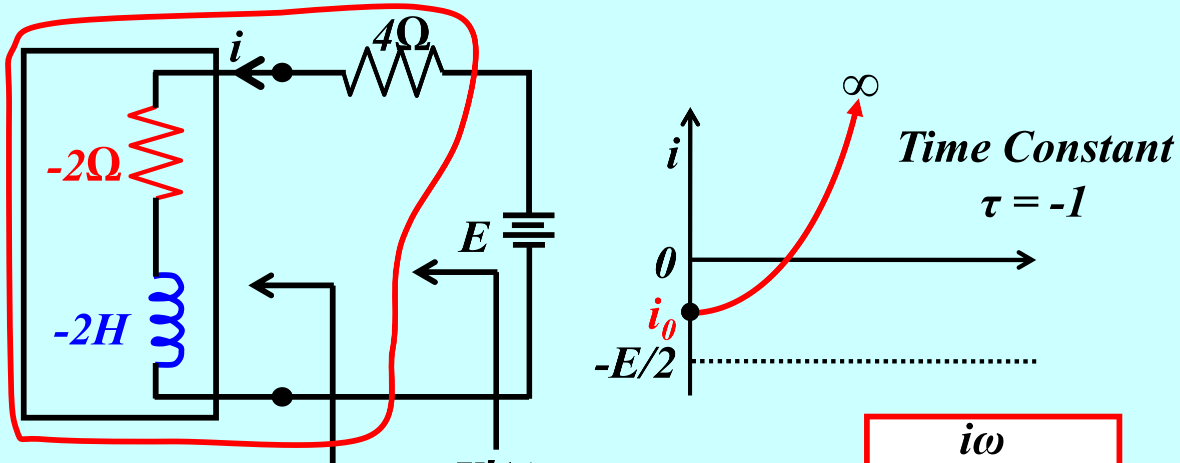


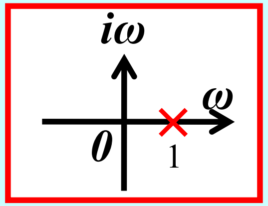


Admittance, $Y(s)=\frac{I(s)}{V(s)}=\frac{1}{(-2-2s)}=\frac{-1}{2(s+1)}$

$$Y(i\omega)=\frac{-1}{2(1+i\omega)}=\frac{-(1-i\omega)}{2(1+i\omega)(1-i\omega)}=\frac{-1}{2(1+\omega^2)}+i\frac{\omega}{2(1+\omega^2)}$$

$$Re\,Y(i\omega)=\frac{-1}{2(1+\omega^2)}<0,\quad -\infty<\omega<\infty$$





Admittance, $Y'(s)=\frac{I(s)}{V(s)}=\frac{1}{(2-2s)}=\frac{-1}{2(s-1)}$

$$Y'(i\omega)=\frac{-1}{2(-1+i\omega)}=\frac{-1(-1-i\omega)}{2(-1+i\omega)(-1-i\omega)}=\frac{1}{2(1+\omega^2)}+i\frac{\omega}{2(1+\omega^2)}$$

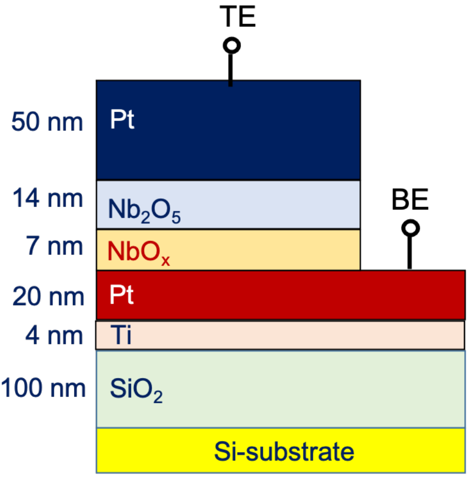
$$Re\,Y'(i\omega)=\frac{1}{2(1+\omega^2)}>0,\quad -\infty<\omega<\infty$$

The voltage-controlled one-port within the black box is poised on the Stable Locally-Active operating regime, also referred to as Edge of Chaos

Including the passive and linear resistor in series with the original one-port, the resulting overall voltage-controlled one-port within the red box is poised on the Unstable Locally-Active operating regime

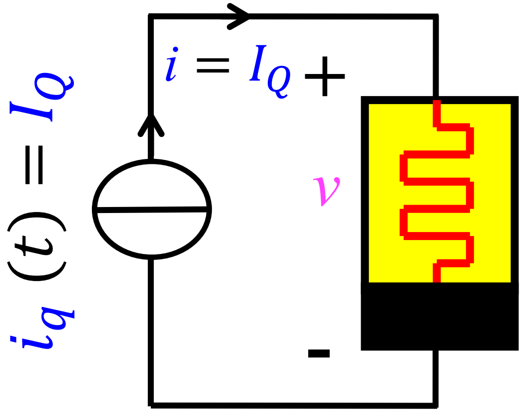
A miniaturized volatile niobium oxide memristor with locally-active behaviour

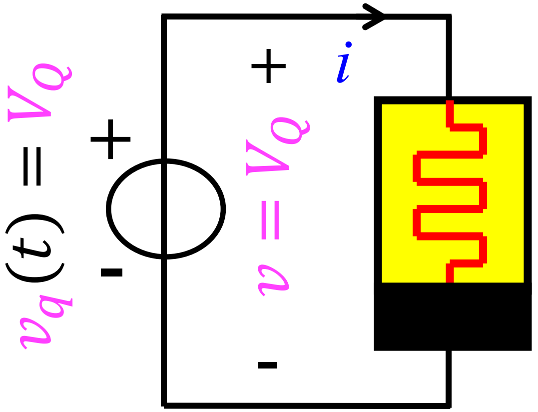
Experimental observations

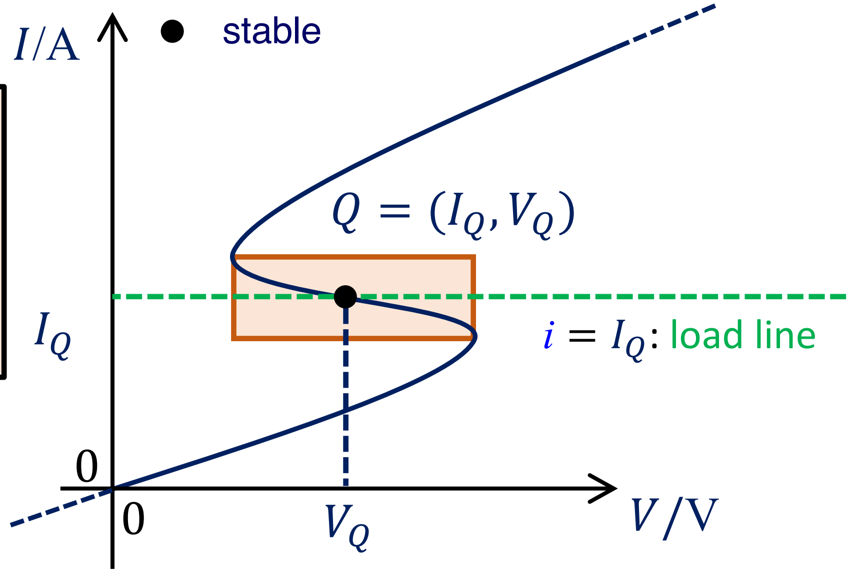


NbO device stack from NaMLab

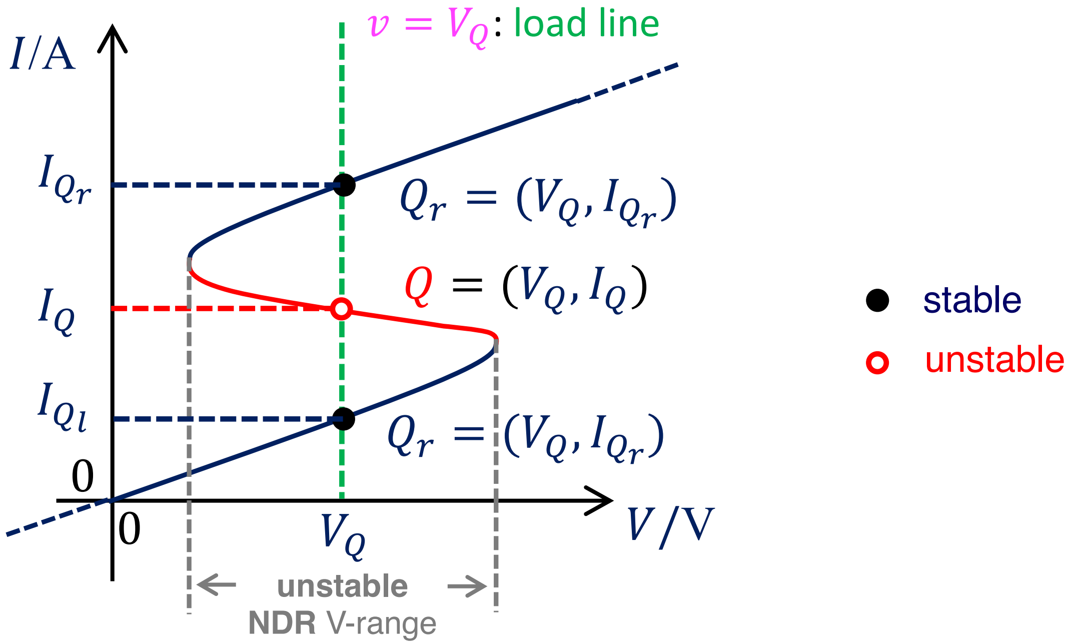
- **stable NDR region**
- **condition 4.** from the Local Activity Theorem **holds true at DC** at **any NDR bias point**







Qualitative sketch of the device DC $V - I$ characteristic obtained under current sweep. **Blue: stable branch**



Qualitative sketch of the device DC $I - V$ characteristic obtained under voltage sweep. Blue: stable branch. Red: unstable branch

NbO_x threshold switch from NaMLab

$$\frac{dx}{dt} = g(x, v)$$

$$i = G(x)v$$

generic memristor model based upon the Unfolding Principle

A. Ascoli, S. Slesazeck, H. Mähne, R. Tetzlaff, and T. Mikolajick, "Nonlinear dynamics of a locally-active memristor," *IEEE Trans. Circuits Systems–I: Reg. Papers*, vol. 62, no. 4, pp. 1165–1175, Apr. 2015

with state evolution function

$$g(x, v) = 5.19 \cdot 10^9 - 2.05 \cdot 10^7 \cdot x$$
$$+ (7.21 \cdot 10^9 - 0.07 \cdot 10^9 \cdot x + 2.27 \cdot 10^5 \cdot x^2 - 2.40 \cdot 10^2 \cdot x^3 + 1.25 \cdot 10^{-1} \cdot x^4 - 2.69 \cdot 10^{-5} \cdot x^5) \cdot v^2$$

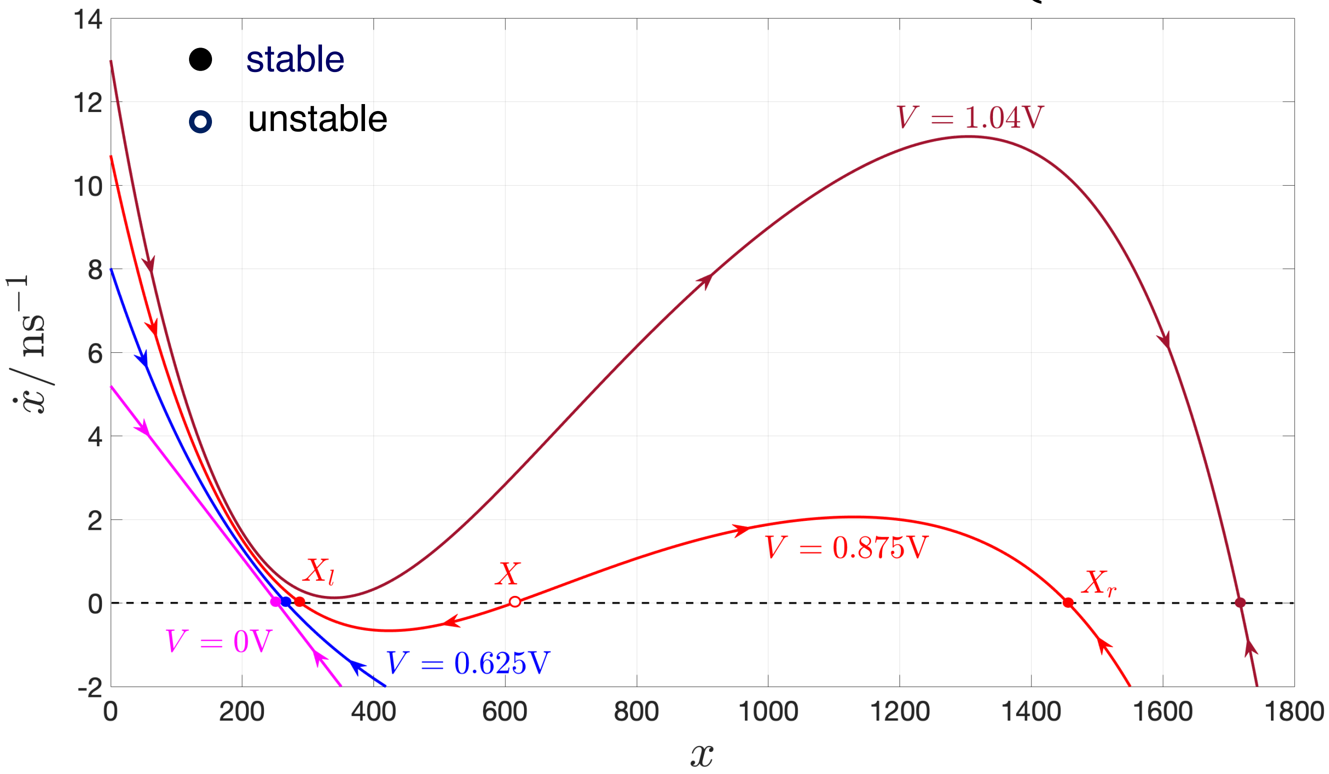
and memductance function

$$G(x) = 6.50 \cdot 10^{-3} - 6.66 \cdot 10^{-5}\, x + 2.14 \cdot 10^{-7} \cdot x^2 - 2.14 \cdot 10^{-10} \cdot x^3 + 1.19 \cdot 10^{-13} \cdot x^4$$

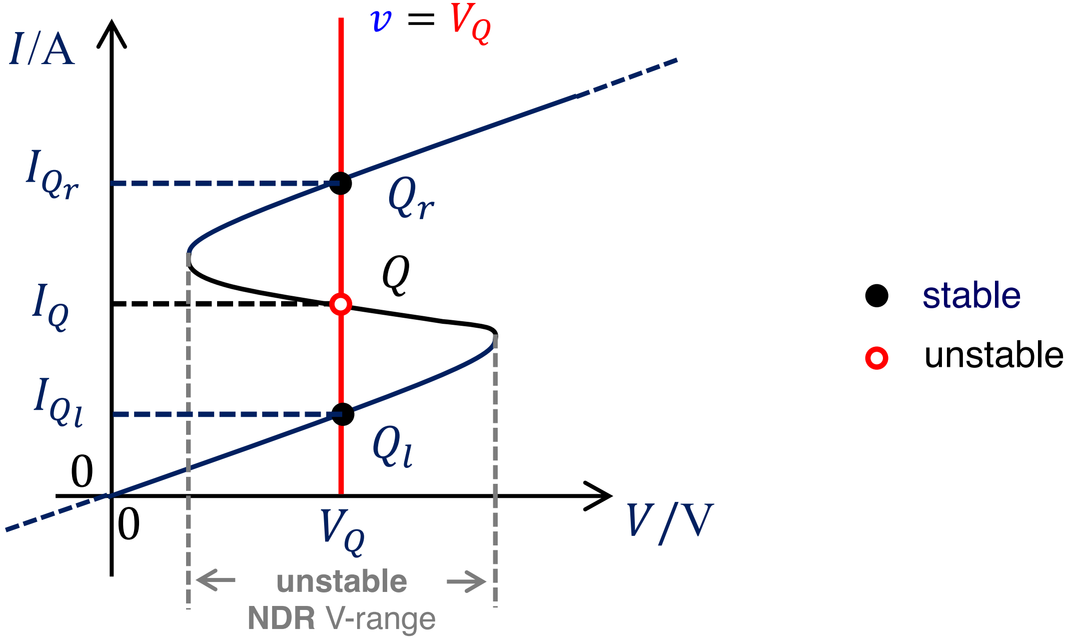
Memristor Dynamic Route Map (DRM) under Voltage Control

Memristor DAE set:

$$\begin{cases} \dfrac{dx}{dt} = g(x, v) \\ i = G(x) \cdot v \end{cases}$$



Memristor DRM under a range of DC voltage values

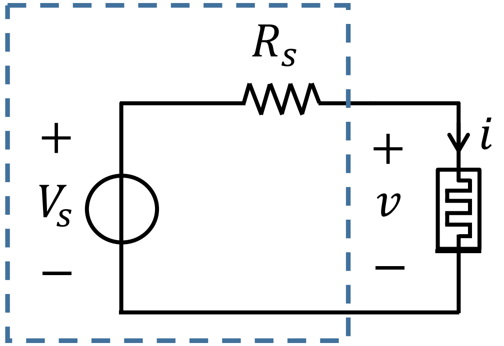


- If the DC voltage lies within the NDR region, there exists <u>3 possible operating points</u>, of which <u>the intermediate NDR one is unstable</u>

A. Ascoli, S. Slesazeck, H. Mähne, R. Tetzlaff, and T. Mikolajick, "Nonlinear dynamics of a locally-active memristor," *IEEE TCAS–I*, vol. 62, no. 4, pp. 1165–1175, 2015

Biasing circuit for stabilizing a NDR operating point on the DC locus of the voltage-controlled device

$$v = \frac{V_S}{1 + G(x) \cdot R_S}$$ memristor voltage



DC biasing circuit [1]

$$\begin{cases} \dfrac{dx}{dt} = g\left(x, \dfrac{V_S}{1 + G(x) \cdot R_S}\right) \\ i = G(x) \cdot \dfrac{V_S}{1 + G(x) \cdot R_S} \end{cases}$$ voltage-controlled memristor model (for the DRM analysis)

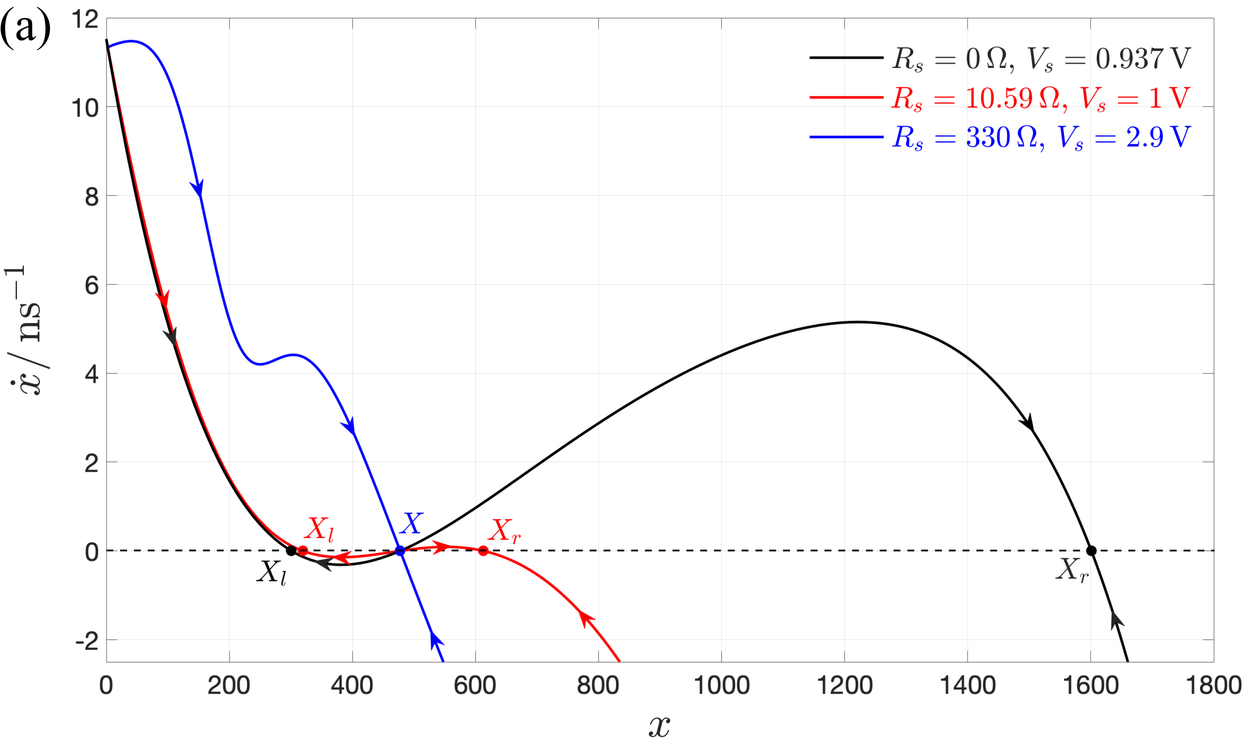
$$i = \frac{V_S - v}{R_S}$$ load line (for determining the intersections with the device DC I versus V locus)

[1] A. Ascoli, S. Slesazeck, H. Mähne, R. Tetzlaff, and T. Mikolajick, “Nonlinear dynamics of a locally-active memristor,” *IEEE TCAS–I*, vol. 62, no. 4, pp. 1165–1175, 2015

Stabilization of an Operating Point on the NDR Region of the DC Characteristic of the Memristor under Voltage Control

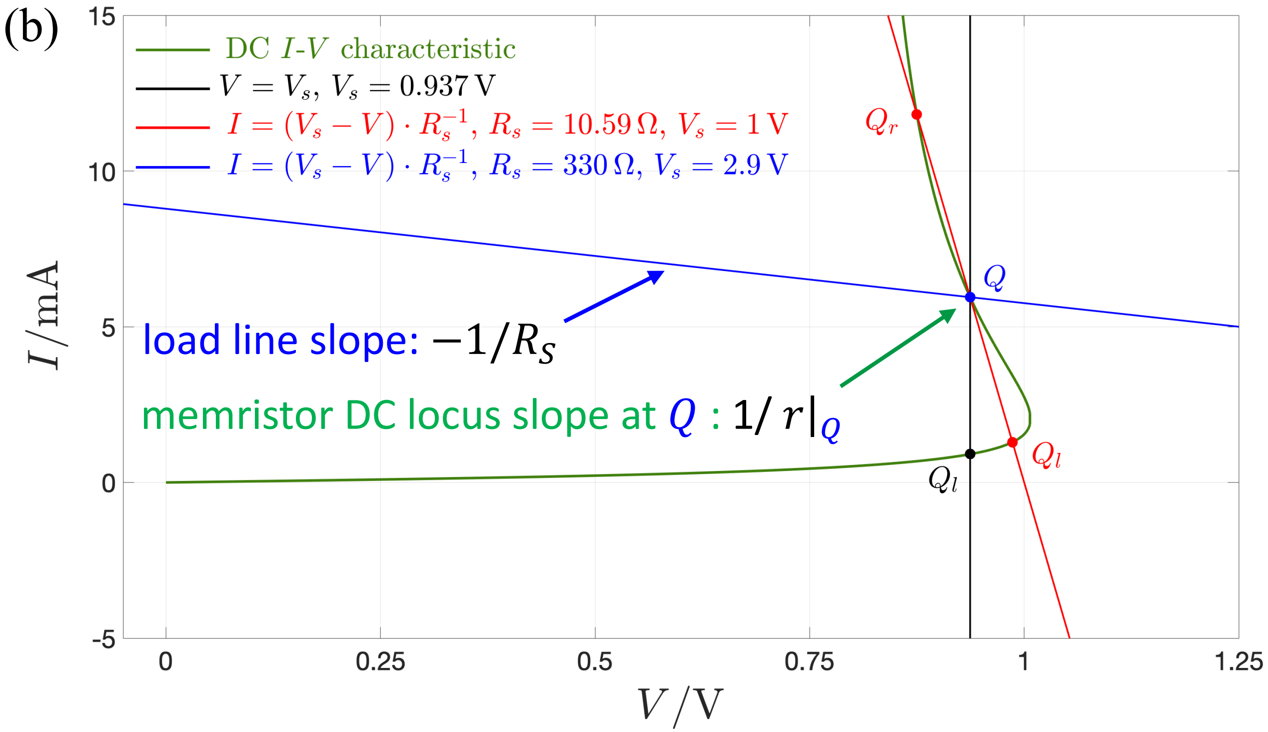
- Apply a DC voltage V_S across the R_S-$\mathcal{M}$ series one-port so that the load line intersects the device DC locus in a NDR operating point Q.

(a)



Arrowed $\frac{dx}{dt} = g\left(x, \frac{V_S}{1+G(x)\cdot R_S}\right)$ versus x loci sharing NDR bias point X

(b)

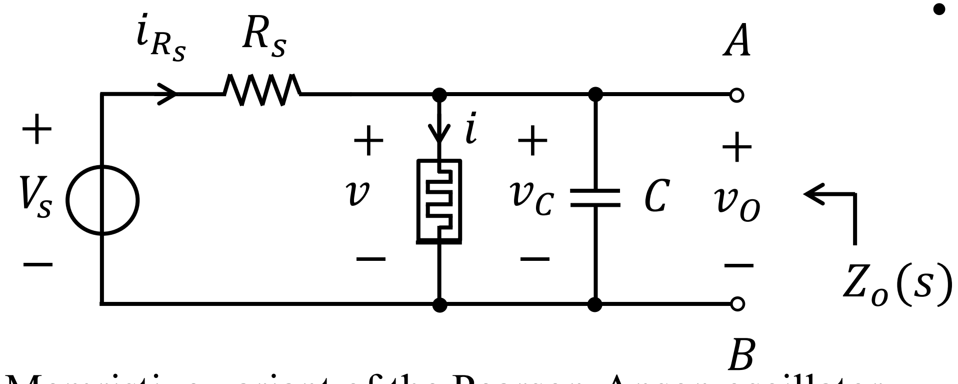


DC load line $I = \frac{V_S - V}{R_S}$ for each (R_S, V_S) pair in (a) and memristor DC locus

$$\boxed{R_S > -r\Big|_Q \triangleq -\frac{1}{\left.\frac{di}{dv}\right|_Q}}$$

condition for the stabilization of NDR operating point Q

Memristive Variant of the Pearson-Anson Relaxation Oscillator and Its Small-Signal Equivalent Circuit Model



Memristive variant of the Pearson-Anson oscillator

State equations of the second-order cell:

$$\begin{cases} \dfrac{dx}{dt} = f_1(x, v) \triangleq g(x, v) \\ \dfrac{dv}{dt} = f_2(x, v) \triangleq \dfrac{1}{C}\left(\dfrac{V_s - v}{R_s} - G(x) \cdot v\right) \end{cases}$$

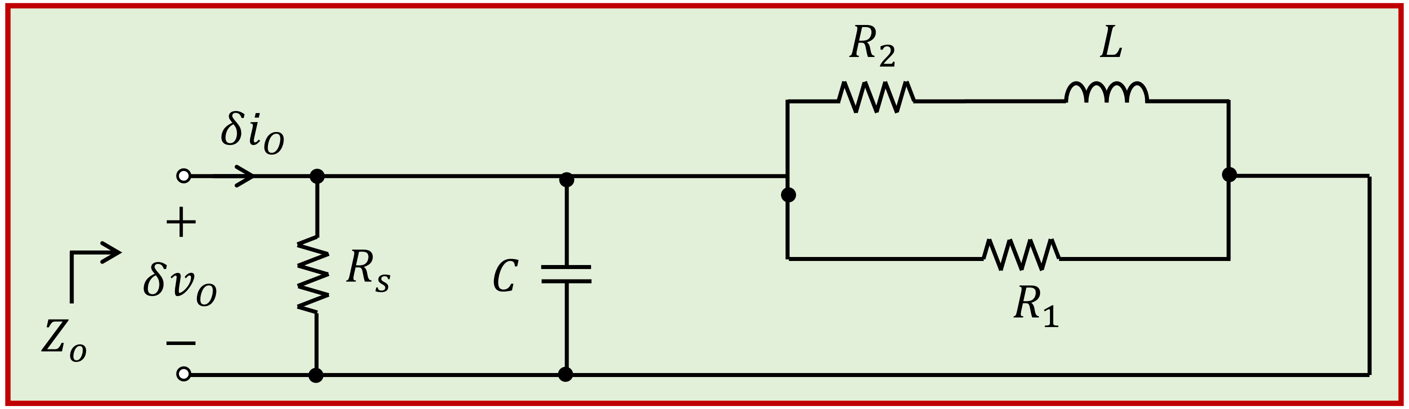
- Local input impedance of the oscillator at the coupling port $A - B$

$$Z_o(s) = \frac{\mathcal{L}\{\delta v_O(t)\}}{\mathcal{L}\{\delta i_O(t)\}} = K \cdot \frac{s - s_{z,Z_o}}{\left(s - s_{p_1,Z_o}\right) \cdot \left(s - s_{p_2,Z_o}\right)}$$

where K, s_{z,Z_o}, and s_{p_i,Z_o} for $i \in \{1,2\}$ may be expressed in terms of the parameters of the memristor small-signal equivalent circuit model as

$$K = \frac{1}{C} \quad , \quad s_{z,Z_o} = -\frac{R_2}{L} \quad ,$$

$$s_{p_i,Z_o} = -\left(\frac{R_2}{L} + \frac{R_1 + R_s}{C \cdot R_1 \cdot R_s}\right) \pm \frac{1}{2} \cdot \sqrt{\left(\frac{R_2}{L} + \frac{R_1 + R_s}{C \cdot R_1 \cdot R_s}\right)^2 - 4 \cdot \frac{1}{L \cdot C}\left(1 + R_2 \cdot \frac{R_1 + R_s}{R_1 \cdot R_s}\right)}$$

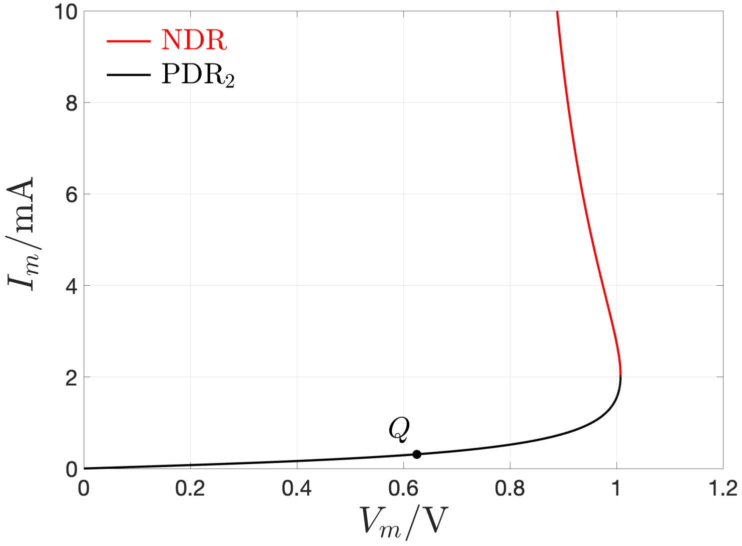


Oscillator small-signal equivalent circuit about $Q_o = (X, V)$ [3]

A. Ascoli, A.S. Demirkol, R. Tetzlaff, and L.O.Chua, "Edge of Chaos Theory Resolves Smale Paradox," *IEEE TCAS–I*, 2022, DOI: 10.1109/TCSI.2021.3133627

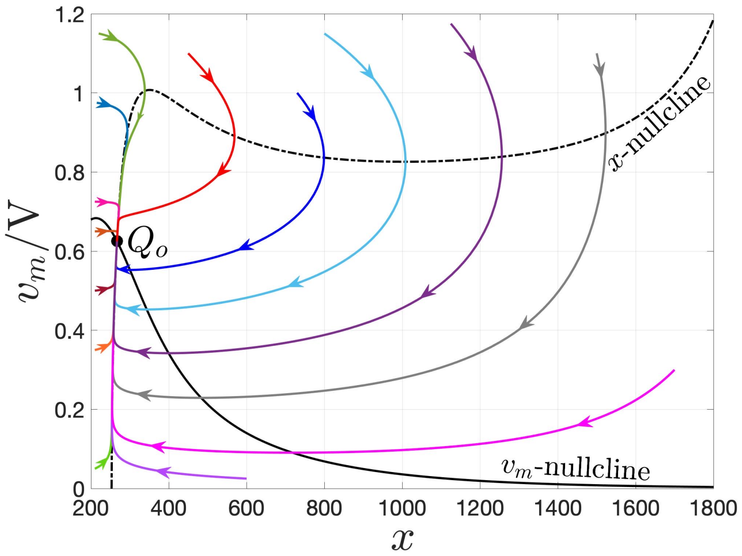
Scenario 1: Cell dynamics when the memristor $\mathcal{M}$ is polarized in one PDR bias point

- Choose V_S and R_S so that the load line meets the device DC I-V locus at a point $Q = (V, I)$ lying on either of the PDR branches
- The cell is found to be locally passive at a globally asymptotically stable (GAS) operating point $Q_o = (X, V)$, irrespective of C



Device bias point for

$V_S = 0.7275$ V, $R_S = 330\ \Omega$



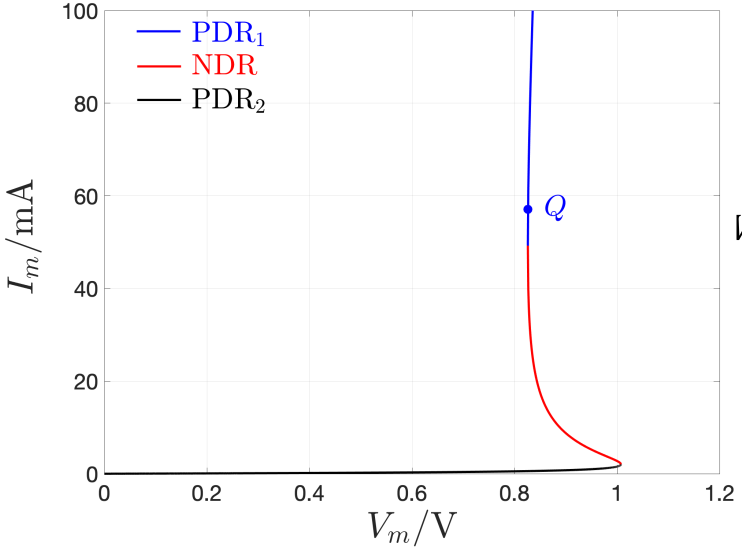
Cell phase portrait for

$V_S = 0.7275$ V, $R_S = 330\ \Omega$, C = 5 nF

---- x-nullcline: $g(x, v) = 0$

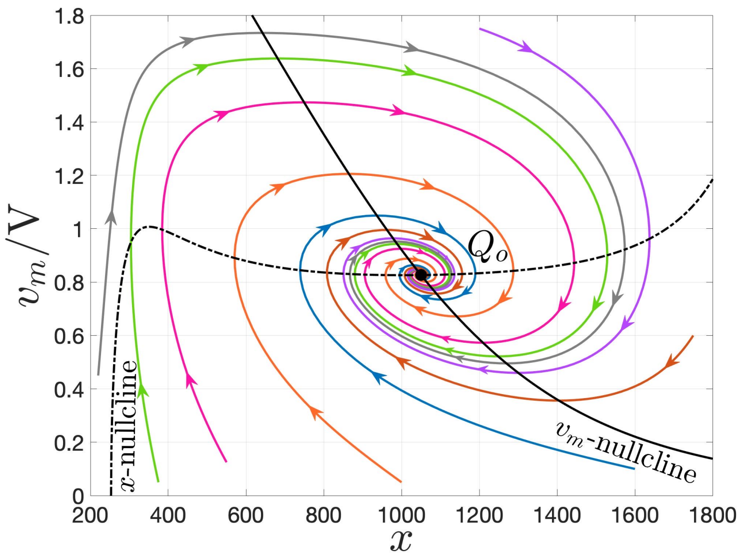
—— v-nullcline: $v = \dfrac{V_S}{1 + G(x) \cdot R_S}$

● stable operating point: $Q_0 = (X, V)$



Device bias point for

$V_S = 2.5392$ V, $R_S = 30\ \Omega$



Cell phase portrait for

$V_S = 2.5392$ V, $R_S = 30\ \Omega$, C = 5 nF

Scenario 2: Cell dynamics when $\mathcal{M}$ may stabilise in either of the PDR regions

- Choose V_S and R_S so that the load line meets the device DC I-V locus in three points, namely

$$Q_l = (V_l, I_l) \text{ in PDR}_1\ , \qquad Q = (V, I) \text{ in NDR}\ , \quad Q_r = (V_r, I_r) \text{ in PDR}_2$$

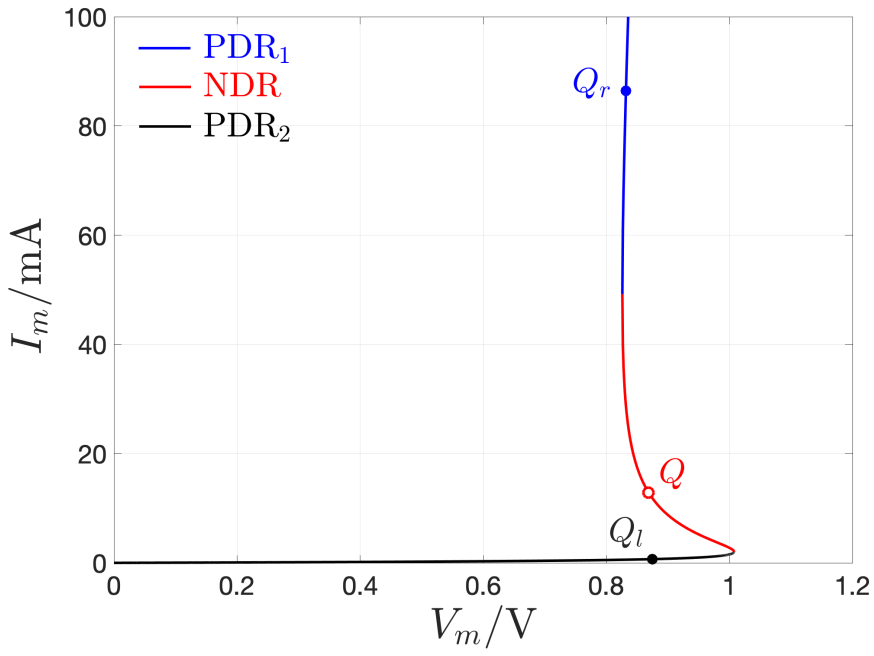
- The cell correspondingly has three operating points

$$Q_{o,1} = (X_l, V_l)\ , \qquad Q_{o,2} = (X, V)\ , \qquad Q_{o,3} = (X_r, V_r)$$

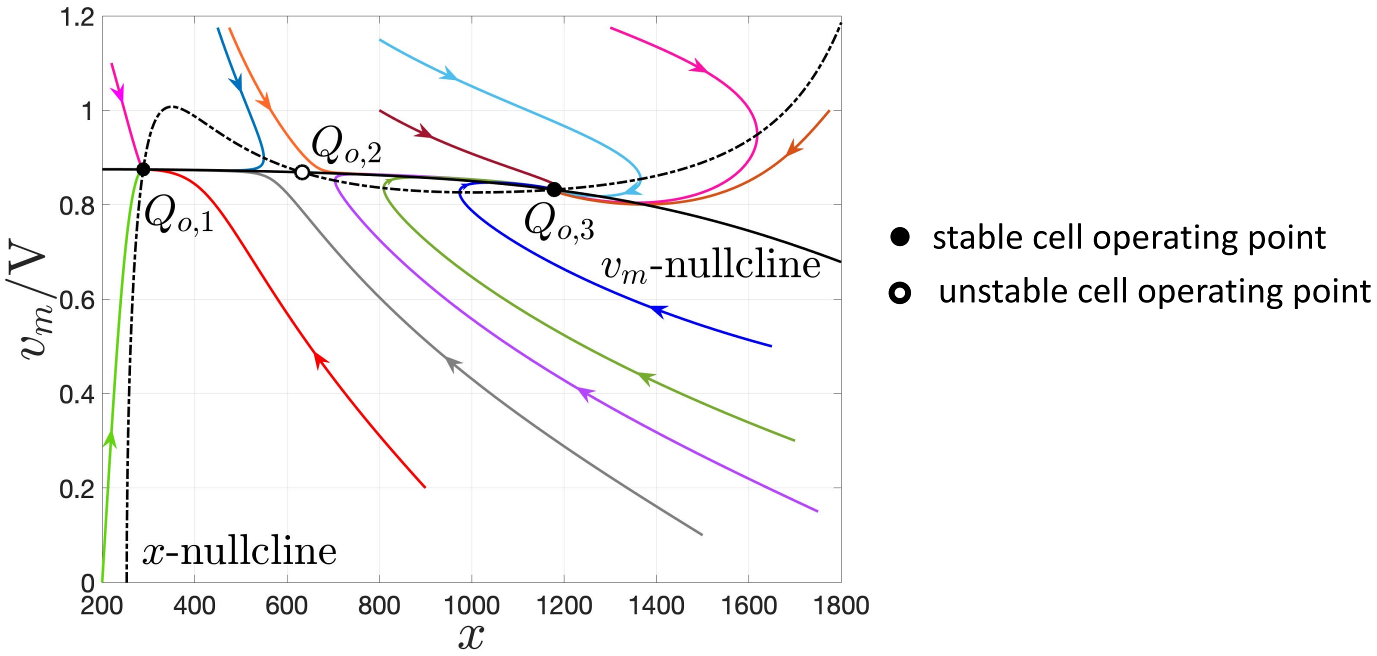
- The cell is found to be

locally passive at either of the locally-stable operating points $Q_{o,1}$ and $Q_{o,3}$, irrespective of C

locally active and unstable at the unstable operating point $Q_{o,2}$, irrespective of C



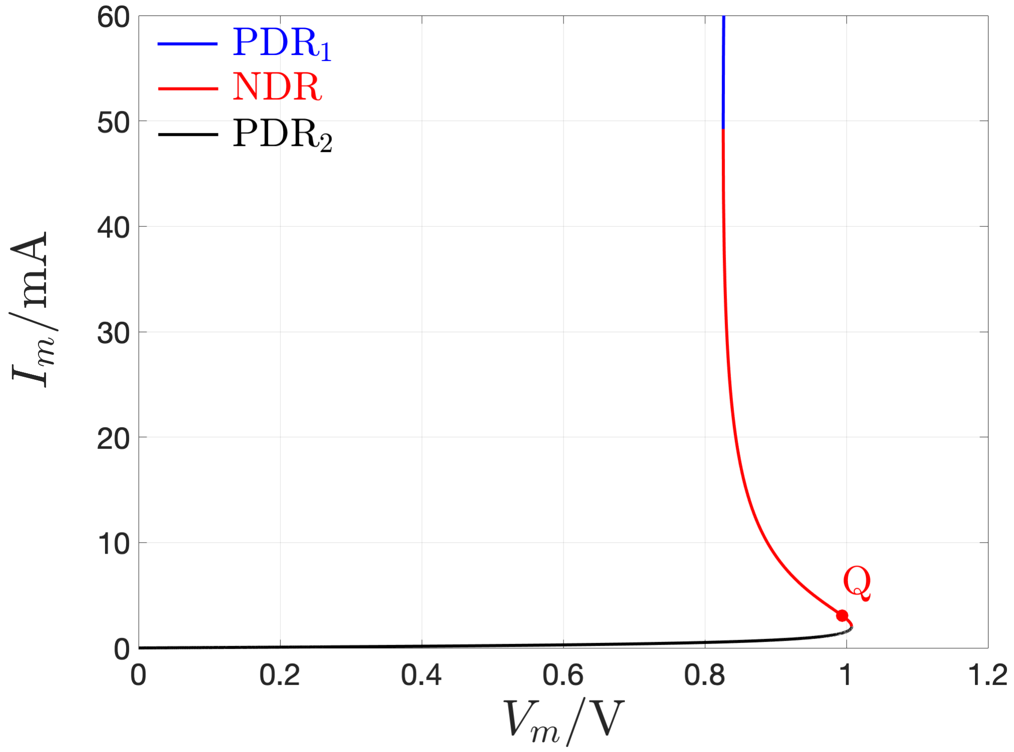
Possible device bias points for $V_S = 0.875$ V, $R_S = 0.5\ \Omega$



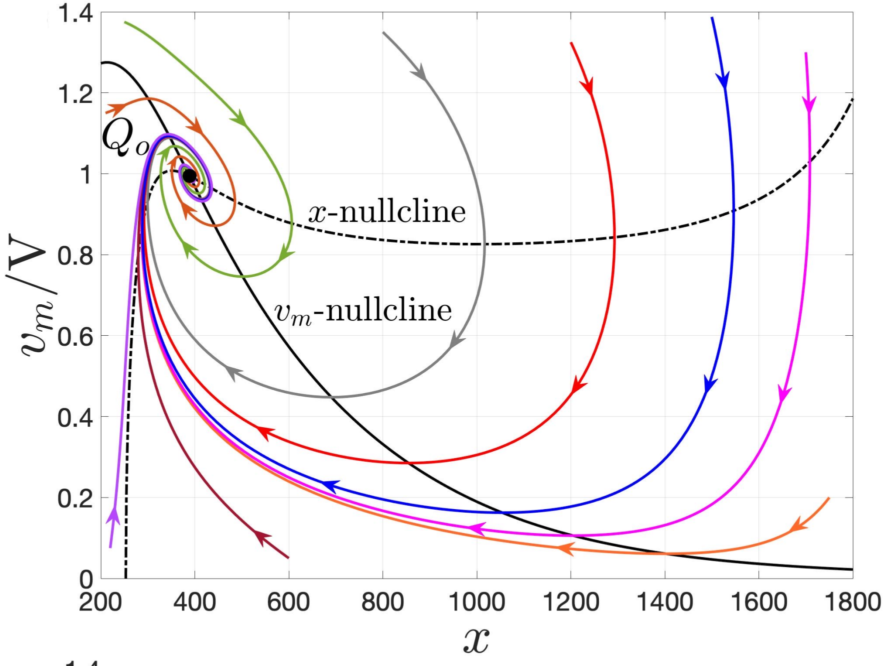
Cell phase portrait for $V_S = 0.875$ V, $R_S = 0.5\ \Omega$, and $C = 100$ nF

Scenario 3: Cell dynamics when $\mathcal{M}$ is polarized in one NDR operating point

- Choose V_S and R_S so that the load line meets the device DC I-V locus at a point $Q = (V, I)$ lying on the NDR
- The stability of the cell the respective operating point $Q_o = (X, V)$ depends critically upon the capacitance C



Device bias point for $V_S = 1.3$ V , $R_S = 100\ \Omega$

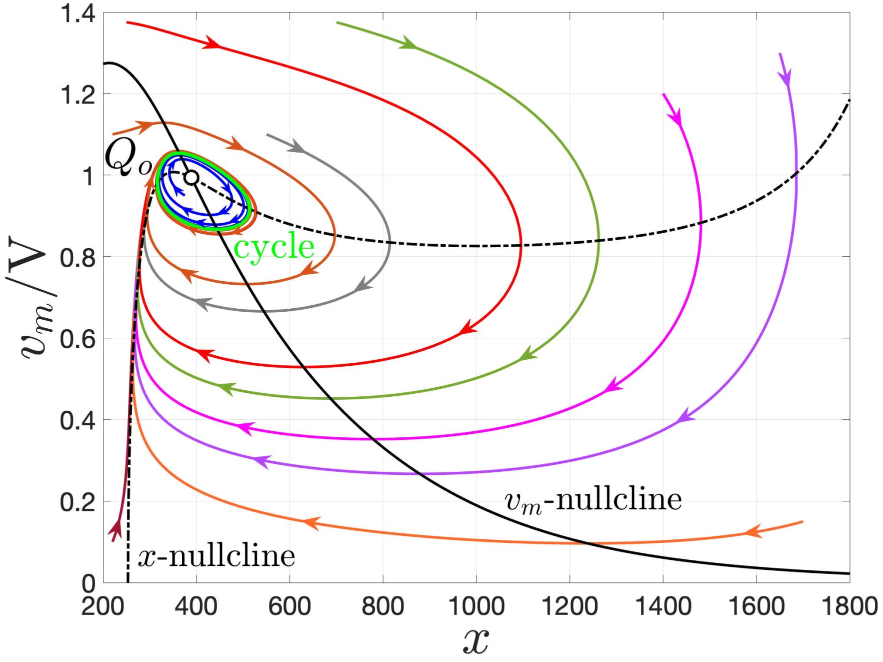


Cell phase portrait for

$V_S = 1.3$ V , $R_S = 100\ \Omega$, C = 2 nF

The cell is silent,
poised on the EOC at Q_o

● stable cell operating point



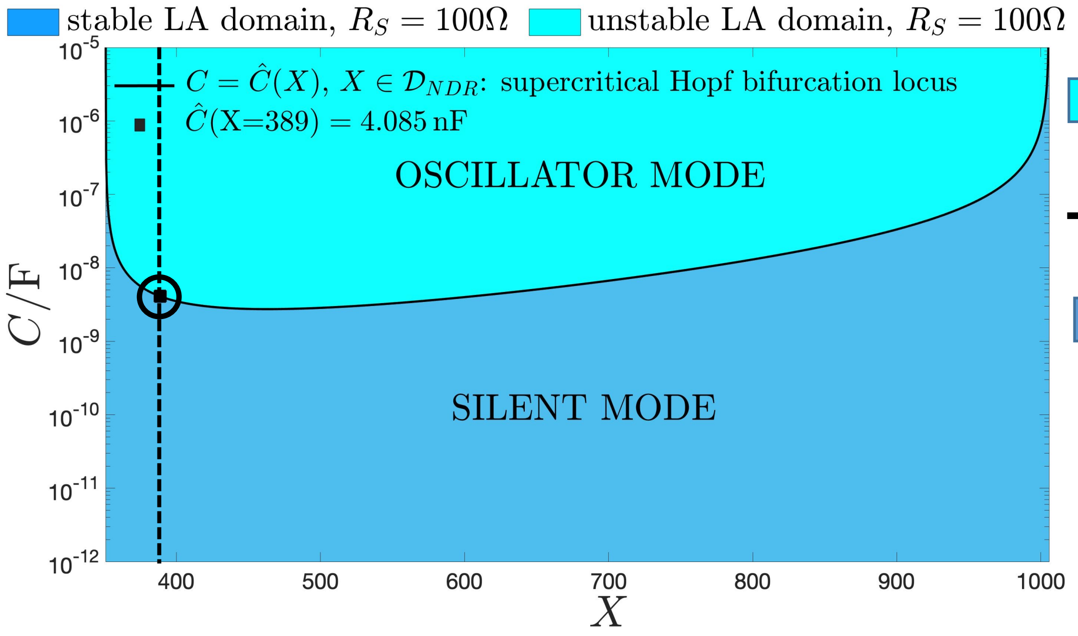
Cell phase portrait for

$V_S = 1.3$ V , $R_S = 100\ \Omega$, C = 6 nF

The cell is an oscillator,
locally active and unstable at Q_o

Classification of Cell Operating Regimes for all Possible Cases Studies in Scenario 3

- Scenario 3: choose V_S and R_S so that $Q = (V, I)$ lies on the NDR branch of the device DC I-V locus.
- Let $R_S = 100\Omega$ $\rightarrow$ the stabilization condition $R_S > -r|_Q$ for the voltage-controlled memristor applies throughout the NDR branch
- The memristor operating point X is directly determined by the choice for V_S



Classification of all possible operating regimes of the cell in scenario 3

for C > $\hat{C}$ the cell is an oscillator, locally active and unstable at $Q_0 = (X, V)$

supercritical Hopf bifurcation locus $C = \hat{C}(X)$

for C < $\hat{C}$ the cell is silent, poised on the EOC at $Q_0 = (X, V)$

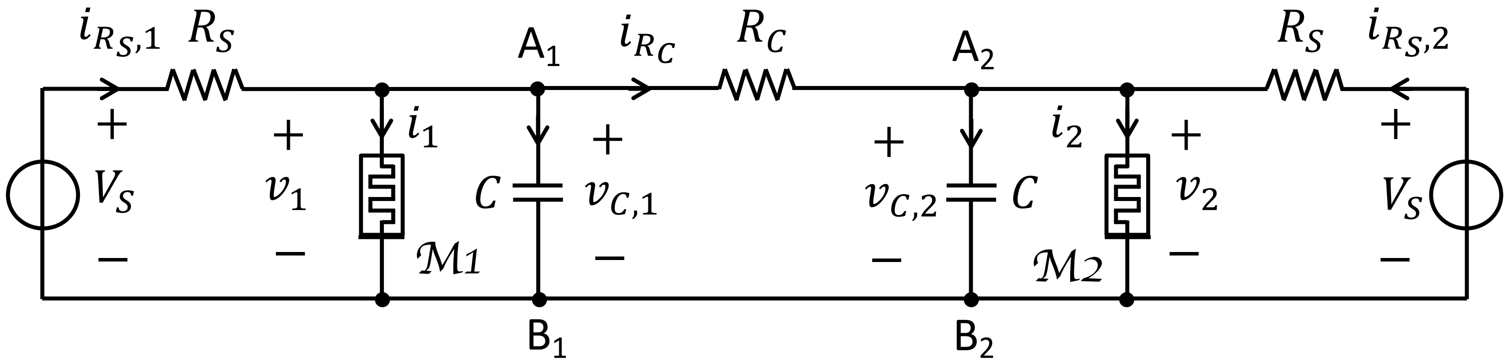
- Case study: if $V_S = 1.3$V, then $X = 389$, $V = 0.994$V

$(X, \hat{C}) = (389, 4.085nF)$ supercritical Hopf bifurcation point

Fundamental Result: Edge of Chaos Theorem

A stable operating point Q of a given one-port may be destabilized by coupling the one port to a passive environment if and only if Q is poised on the ***Edge of Chaos***

Coupled System



- Two identical cells, poised on the EOC on their own, are diffusively coupled through a passive and linear resistor R_C
- State equations:

$$\frac{dx_1}{dt} = f_1(x_1, v_1, x_2, v_2) \triangleq g(x_1, v_1)$$

$$\frac{dv_1}{dt} = f_2(x_1, v_1, x_2, v_2) \triangleq \frac{1}{C}\left(\frac{V_s - v_1}{R_s} - G(x_1) \cdot v_1 + \frac{v_2 - v_1}{R_C}\right)$$

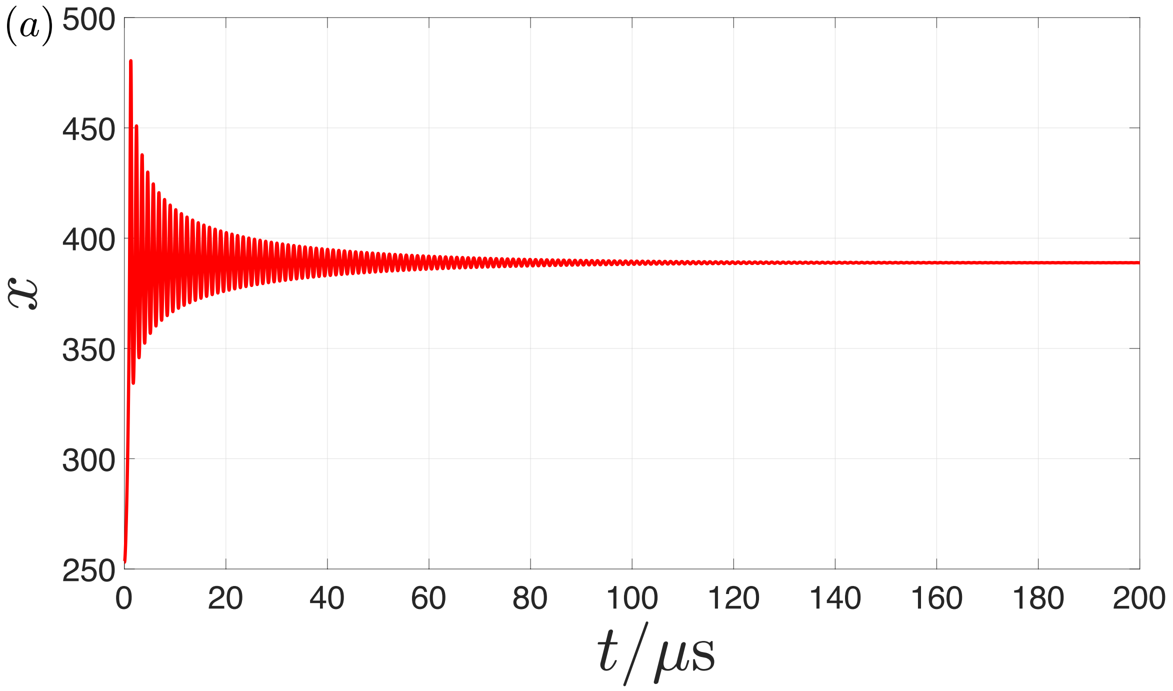
$$\frac{dx_2}{dt} = f_3(x_1, v_1, x_2, v_2) \triangleq= g(x_2, v_2)$$

$$\frac{dv_2}{dt} = f_4(x_1, v_1, x_2, v_2) \triangleq \frac{1}{C}\left(\frac{V_s - v_2}{R_s} - G(x_2) \cdot v_2 + \frac{v_1 - v_2}{R_C}\right)$$

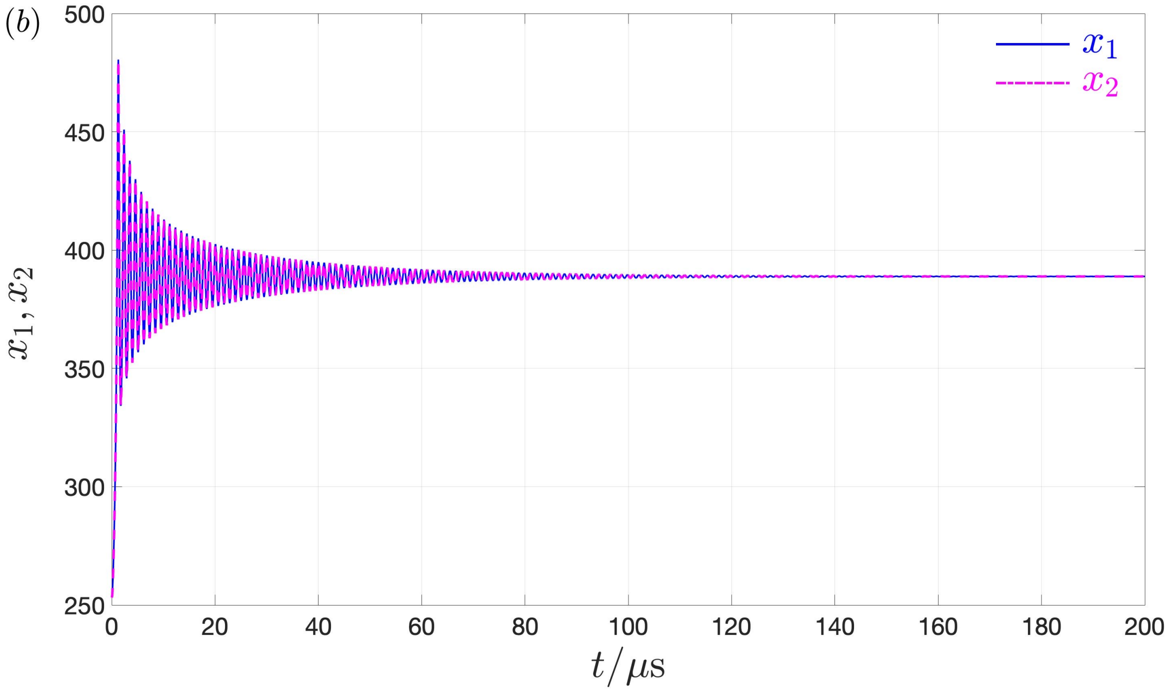
- The common expectation is that, irrespective of R_C, the memristive array would admit the homogeneous solution, where each of the two identical cells converges to the GAS operating point it would approach in the uncoupled case.
- Surprisingly, this is the case only for appropriately large R_C values.

Uncoupled cell under silence, and Homogeneous Solution of the Two-Cell Array

- Uncoupled cell, memristor bias parameters: $R_S = 100\Omega$, and $V_S = 1.3$V.
- For $\mathrm{C} = 4$nF $< \hat{C} = 4.085$nF the uncoupled cell is poised on the EOC domain
- Coupling two identical copies of this cell via a resistor of large resistance, the resulting array displays the homogeneous solution



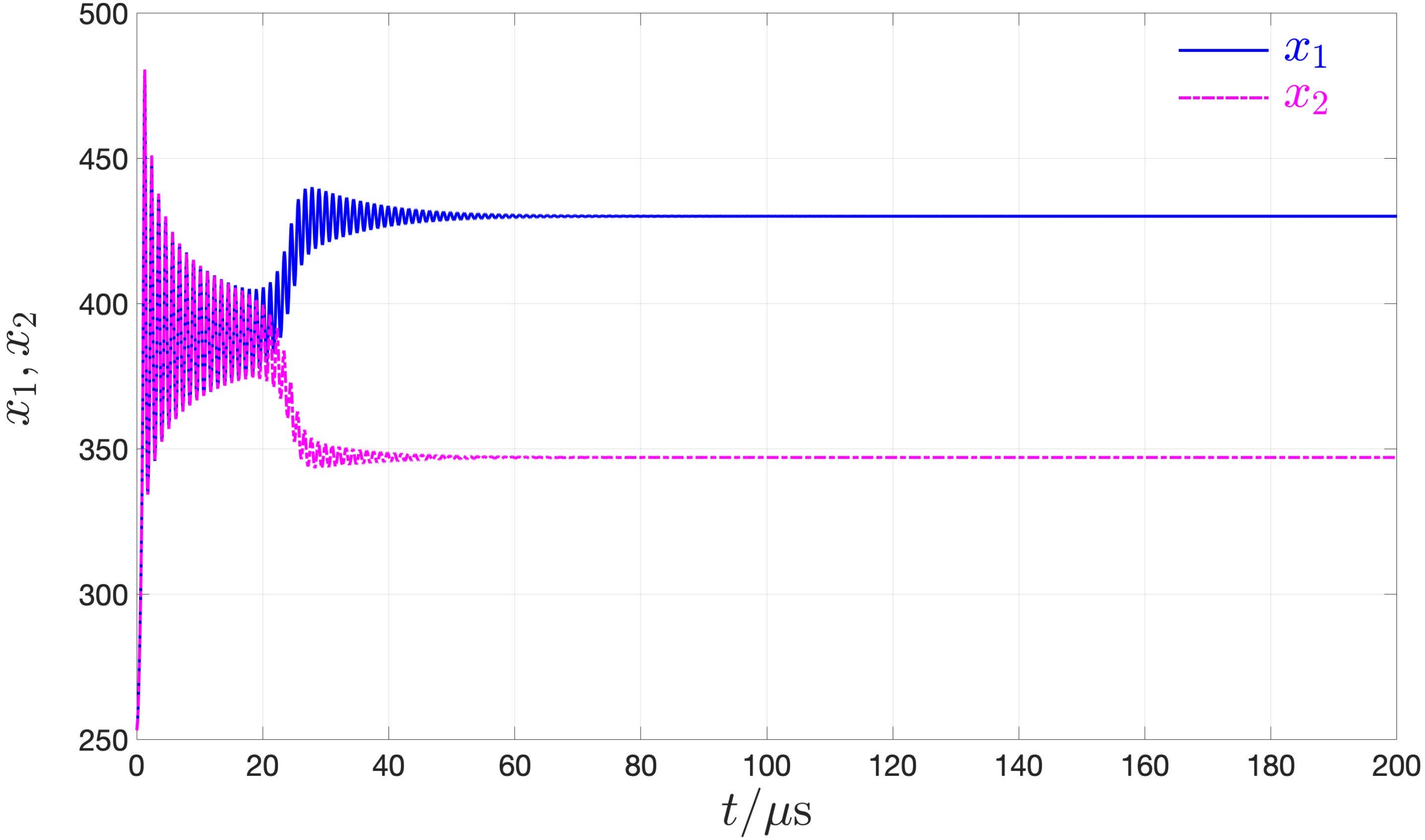
Single cell, approaching a globally asymptotically stable operating point (silent state) as times goes to infinity



Homogeneous solution of the two-cell array. Here $R_c = 50\Omega$

Diffusion-driven Instabilities in the Two-Cell Array:
Formation of Static Turing patterns

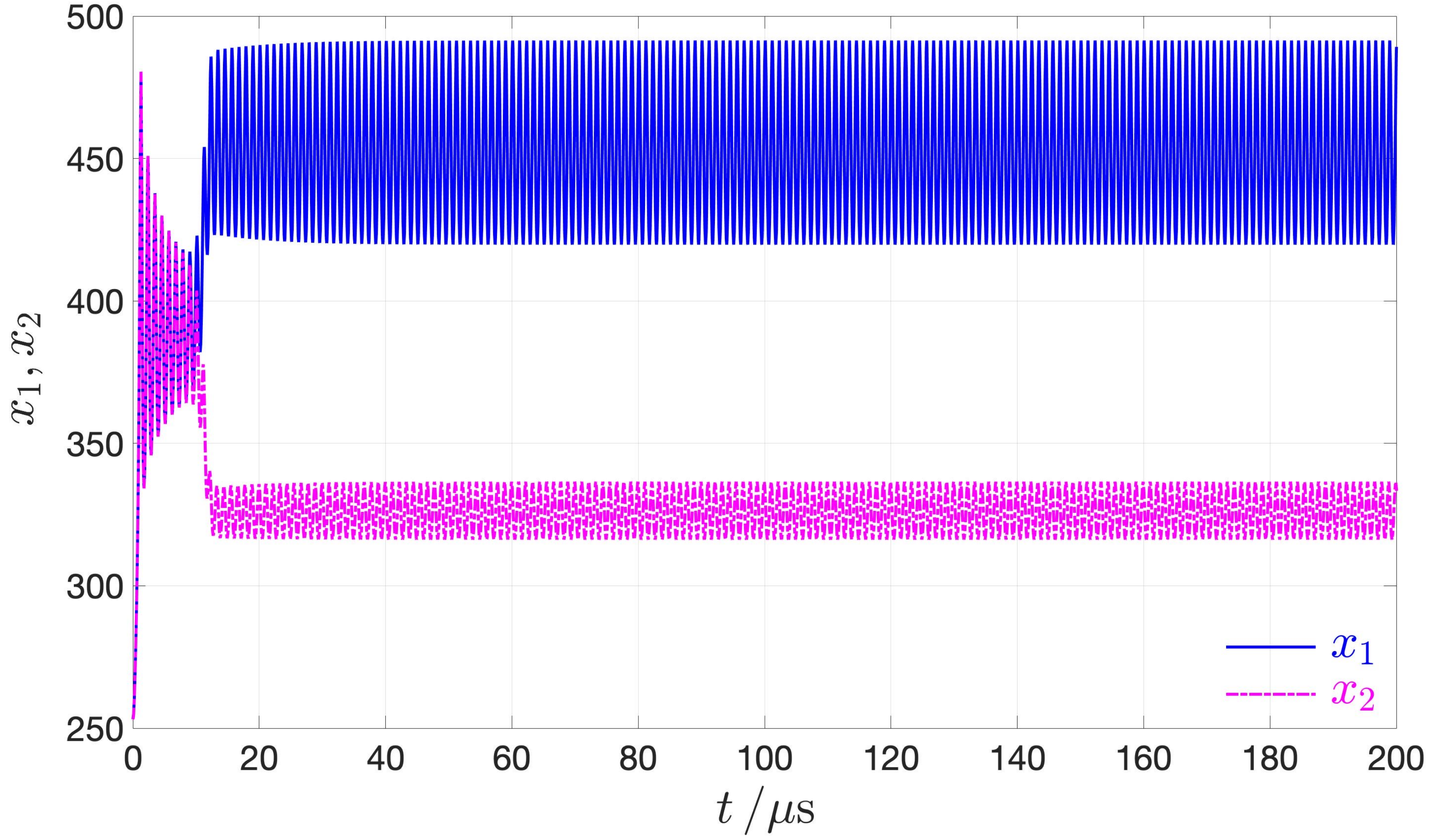
- The destabilisation of the homogeneous solution, due to a pitchfork bifurcation, gives way to a Turing pattern for $R_C = 49.7\Omega$



Development of an inhomogeneous static solution, i.e. a Turing pattern, in the two-cell array for $R_C = 40\Omega$

Diffusion-driven Instabilities in the Two-Cell Array: Formation of Dynamic Patterns

- Decreasing the coupling resistance further, oscillatory waveforms first develop in the cellular medium out of a Hopf supercritical bifurcation, at the expenses of the inhomogeneous static solution, for $R_C = 28.1\ \Omega$



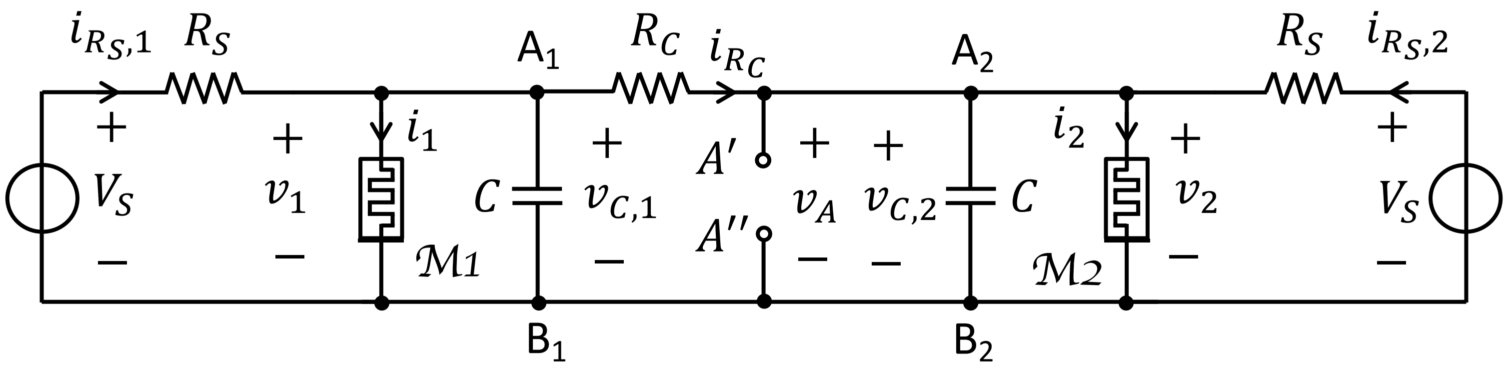
Development of an inhomogeneous dynamic solution in the two-cell array for $R_C = 25\Omega$

Smale, 1974:

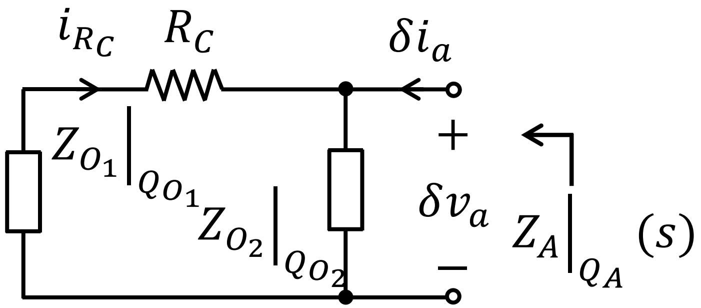
"There is a paradoxical aspect to the example. One has two dead (mathematically dead) cells interacting by a diffusion process which has a tendency in itself to equalize the concentrations. Yet in interaction, a state continues to pulse indefinitely."

A simple 4^{th}-order bio-inspired reaction-diffusion memristor cellular array reproduces the same paradoxical phenomena observed by Smale in a 8^{th} order system from cell biology

Some Insights on the Bifurcations of the Bio-Inspired Array



The simplest ever reported electrical circuit reproducing Smale paradoxical observations



Local equivalent circuit model of the two-cell array

- The closed-form expression for the small-signal impedance of the memristor array is

$$Z_a\Big|_{Q_a}(s) = \frac{\mathcal{L}\{\delta v_A(t)\}}{\mathcal{L}\{\delta i_A(t)\}} = Z_{o_2}\Big|_{Q_{o_2}}(s) \;\; || \;\; \left(R_C + Z_{o_1}\Big|_{Q_{o_1}}(s) \right)$$

where $Q_A = (Q_{O_1}, Q_{O_2})$, with $Q_{O_1} = (X_1, V_1) \neq Q_{O_2} = (X_2, V_2)$

- As R_C is decreased, a 1st bifurcation occurs when 1 of the 4 poles of $Z_A|_{Q_A}$ for $Q_{O_1} \equiv Q_{O_2} = (X, V)$ moves to the RHP, i.e. for [3]

$$R_C = R_{C,1} \triangleq \frac{-2 \cdot r_2(X)}{1 + \dfrac{r_2(X)}{r_1(X) \;||\; R}}$$

For our case study, $R_{C,1} = 49.7\ \Omega$, as observed earlier, the first time a Turing pattern forms in the bio-inspired array.

- As R_C is decreased further, a 2nd bifurcation occurs at $R_C = R_{C,2}$, when a complex conjugate pole pair of $Z_A|_{Q_A}$ for $Q_{O_1} \neq Q_{O_2}$ move to the RHP

Using a numerical method to track the evolution of the poles of $Z_A|_{Q_A}$ on complex plane, the theory predicts a value 28.1 Ω for $R_{C,2}$, as observed earlier, the first time the cells were first found to pulse together, forming a dynamic pattern, in numerical simulations.

A. Ascoli, A.S. Demirkol, R. Tetzlaff, and L.O.Chua, "Edge of Chaos Theory Explains Smale Paradox," *TCAS-I*, 2022, DOI: 10.1109/TCSI.2021.3133627

Edge of Chaos Theory
Explains the
Origin
for the
Diffusion-driven
Instabilities
in the Two-Cell Array

Conclusions

- Cellular Neural/Nonlinear Networks with Locally-Active Memristors may allow
  1. to process data more efficiently than conventional purely-CMOS structures
  2. to reproduce complex phenomena in biological systems, including the human brain
- In this seminar we presented the simplest ever-reported reaction-diffusion system supporting the Smale Paradox, explaining, once and for all, the mechanisms behind diffusion-driven static and dynamic pattern formation, therein
- Edge of Chaos may be interpreted as a new physics principle, which extends the second law of thermodynamics to open systems
- This principle explains the hidden mechanisms underlying the emergence of heterogeneous patterns in homogeneous media, what Prigogine defined as the instability of the homogeneous
- With an outlook toward future research, the theory of Local Activity shall enable the development of a systematic and rigorous approach to design bio-inspired circuits with small-signal memristive amplifiers
- Applications include the development of high-performance brain-like machines and biologically-plausible neuromorphic systems

Thank You

A. Ascoli, S. Slesazeck, H. Mähne, R. Tetzlaff, and T. Mikolajick, “Nonlinear dynamics of a locally-active memristor,” *IEEE TCAS–I*, vol. 62, no. 4, pp. 1165–1175, 2015

A. Ascoli, A.S. Demirkol, R. Tetzlaff, S. Slesazeck, T. Mikolajick, and L.O. Chua, “On Local Activity and Edge of Chaos in a NaMLab Memristor”, *Frontiers in Neuroscience*, 2021, DOI: 10.3389/fnins.2021.651452

A. Ascoli, A.S. Demirkol, R. Tetzlaff, and L.O.Chua, “Edge of Chaos Theory Explains Smale Paradox,” *TCAS-I*, 2022, DOI: 10.1109/TCSI.2021.3133627

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