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ALFRED FETTWEIS

Electrical communications, fluid dynamics, and some fundamental issues in physics

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Content

Electrical communications, fluid dynamics, and some fundamental issues in physics[2]

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Abstract

Strict validity of Maxwell's equations in vacuum is assumed and concepts such as *field velocity*, *rest field* , *autonomous* (charge and current densities seen as properties, not sources), and *basal electromagnetic (EM) field* are introduced. *Flow equations* are derived that for a basal EM field are structured exactly like those of fluid dynamics and thus in fact describe an *EM fluid*. Condensed field configurations, in particular of rotating type, are analysed whose properties make them well suited to serve as models for an electron (positron) and a photon, respectively. Although these fields are essentially contained in a very small volume, they are nowhere point-like, neither in position, nor in time, nor in frequency.

At their *basic (first) level of observation*, electrons and photons then appear to consist of an EM fluid; its detailed behaviour is governed by the laws of the alternative relativistic dynamics published in recent years. At a *second* level of observation, the focus is directed on energy density, which changes indeed due to two distinct effects: convection and work done by the internal forces. The combined overall energy migration can be described by a single energy velocity, and the resulting expressions for mass and energy densities agree with the laws of classical relativistic dynamics. At the *third* level of observation, finally, the movement of a particle as a whole is observed, thus ignoring the details of the behaviour inside of the EM fluid. This movement turns out to be governed by the laws of classical relativistic dynamics, which is in full agreement with known experimental results. Using the electron model, the determination of Sommerfeld's fine-structure constant, can, in principle, be reduced to a purely mathematical problem. The photon-like model exhibits all 13 generally known photon properties and offers a natural in-

[2] This is a more complete text about the results presented by the author in a lecture on 8 July 2009 at the Nordrhein-Westfälische Akademie der Wissenschaften und der Künste in Düsseldorf, Germany.

terpretation of the wave-particle duality. All results follow by strict mathematical deduction from Maxwell's equations and their relativistic transformation rules. No incompatibility arises between relativity theory and the quantum principle.

Nachrichtentechnik, Fluiddynamik und einige Grundfragen der Physik[3]

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Zusammenfassung

Grundannahme ist die strenge Gültigkeit der Maxwellschen Gleichungen im Vaku-um. Konzepte wie *Feldgeschwindigkeit*, *Ruhfeld*, *autonomes* (Ladungs- und Stromdichten aufgefasst als Eigenschaften, nicht Quellen) und *basales elektromagnetisches* (*EM*) *Feld* werden eingeführt. *Strömungsgleichungen* werden hergeleitet, die für ein basales Feld die gleiche Struktur besitzen wie die Grundgleichungen der Fluiddynamik, also eigentlich ein *EM-Fluid* beschreiben. Verdichtete Felder, insbesondere von rotierendem Typ, werden untersucht, die als Modelle für ein Elektron (Positron) bzw. ein Photon geeignet sind. Obwohl solche Felder auf ein sehr kleines Volumen beschränkt sind, sind sie keinesfalls einfach punktförmig, und zwar weder im Ort, noch in der Zeit, noch in der Frequenz.

Auf der *primären Beobachtungsebene* erweisen sich Elektronen und Photonen damit als EM-Fluide, deren Detailverhalten durch die vor einigen Jahren bekannt gewordene alternative relativistische Dynamik beschrieben wird. Energiedichten ändern sich infolge zweier sehr unterschiedlicher Effekte: Konvektion und durch die Kräfte geleistete Arbeit. Beide Effekte zusammen ergeben eine *Energiemigration*, die mittels einer resul-

[3] Ausführlichere Darstellung der Ergebnisse, über die am 8. Juli 2009 unter dem Titel "Mechanistische Eigenschaften elektromagnetischer Felder, nachrichtentechnische Anforderungen und einige Grundfragen der Physik" auf einer gemeinsamen Sitzung der Klasse für Naturwissenschaften und Medizin und der Klasse für Ingenieur- und Wirtschaftswissenschaften der Nordrhein-Westfälischen Akademie der Wissenschaften und der Künste in Düsseldorf berichtet worden ist.

tierenden *Energiegeschwindigkeit* erfasst werden kann (*sekundäre* Beobachtungsebene) und zu Ausdrücken für Massen- und Energiedichten führt, die wie in der klassischen relativistischen Dynamik aufgebaut sind. Entsprechend genügen auch die dynamischen Gleichungen eines geladenen EM-Teilchen, das sich als Ganzes bewegt *(tertiäre* Beobachtungsebene), der klassischen relativistischen Dynamik, in Übereinstimmung mit den bekannten experimentellen Ergebnissen. Aus dem Elektron-Modell lässt sich – prinzipiell - die Bestimmung der Sommerfeldschen Feinstrukturkonstante auf eine rein mathematische Aufgabe zurückführen. Das Photonmodell besitzt alle 13 üblicherweise bekannten Eigenschaften eines Photons und bietet eine natürliche Interpretation des Welle-Teilchen-Dualismus. Alle Ergebnisse folgen streng mathematisch aus den Maxwellschen Gleichungen und deren relativistischen Transformationsregeln. Inkompatibilitäten zwischen Auffassungen der Relativitäts- und der Quantentheorie treten nicht auf.

1. Introduction

This paper addresses some fundamental issues in physics but from a point of view inspired by results from the theories of communications systems and fluid dynamics. In line with this it assumes the strict validity of Maxwell's equations in vacuum, even down to the smallest dimensions, and their inherent relativistic transformation rules.

In an attempt to finding a proper mechanistic interpretation of Maxwell's theory, the concepts of *field velocity* and *rest field* of an electromagnetic (EM) field are introduced and precisely defined. For this, use is made of the known equations for determining the new field quantities after subjecting the coordinates (position coordinates x, y, z and time t) to a Lorentz transformation (Section 2). The judiciousness of these definitions is subsequently confirmed by deriving a variety of results and by showing several of their consequences (Sections 2 and 3). Using rigorous mathematical deductions from Maxwell's equations, the known equations involving field momentum and stress tensor as well as the equation relating Poynting vector and energy are then found to be equivalent to new equations, say *flow equations*, that are of the same type as the corresponding equations of fluid dynamics (Section 4) and thus lend themselves indeed to a consistent mechanistic interpretation. We speak about this as an *observation* at the *primary* or *basic level*. At this basic level, the EM field thus behaves like an *electromagnetic fluid* (*EM fluid*) that is moving under the influence of the surface and volume forces acting in it. In particular, changes of energy densities are caused by to two entirely distinct effects, convection and work done by these forces.

On the other hand, one may be interested in the combined result of these two effects and characterize the overall energy migration simply by means of an *effective energy velocity*. We refer to this as an *observation* at the *secondary level*. The effective energy velocity reaches twice the field velocity at the low end and becomes equal to it at the speed of light. Whether at the primary or the secondary level, the flow equations clearly show that an EM field inherently has inertia and thus mass.

At the secondary level, thus when using the effective energy velocity, the original flow equations admit a form that exhibits relevant properties, at least in as far as the expressions for mass and energy are concerned, that are in agreement with classical relativistic dynamics [1-12]. At the primary level, however, the results are better compatible with

those of an alternative relativistic dynamics that was first mentioned in [13, 14] and had then gradually been refined in a sequence of papers, the last one of which being [15]; it is summarized and partly corrected in the present Appendix C. Even at the secondary level, this alternative theory still plays a role and may therefore not simply be replaced by the classical theory. It had actually been inspired by earlier work on topics in circuit theory, digital signal processing, and on a robust, physically-motivated approach, based on wave-digital concepts [16-18], for numerically integrating partial differential equations representing physically meaningful systems. Nevertheless, as interesting and instructive as these interpretations at the primary and secondary level may be, their details do not affect the validity of the subsequent developments, thus of the bulk of the results presented in this text.

Requiring a current to be a consequence of moving charges leads to the concept of an *autonomous* (self-sustaining) *field* and, more specifically, of a *basal field*, i.e., an autonomous field described in a reference system where its rest current vanishes everywhere (Sections 2 and 4). For basal fields the flow equations become particularly elegant. All this makes it quite plausible that elementary particles, at least EM particles such as electrons, positrons, and photons, are nowhere point objects but condensed fields. In accordance with what has been said above, they would have an inner structure for which, on the one hand, Maxwell's theory still strictly holds but which, on the other, may be considered to consist of an EM fluid. In that fluid, the details of the flow of momentum, energy and charge densities are observed at the basic level and hence follow laws of the alternative relativistic dynamics. At the secondary level, however, classical relativistic dynamics also plays an important role. At the *tertiary level*, finally, i.e. if the fine details are ignored and only the movement of a particle as a whole is observed, it is classical relativistic dynamics that determines the overall behaviour (Section 7), confirming the excellent agreement of all major experiments with the predictions of that theory.

The relevance of such considerations is confirmed by examining two specific localized basal EM fields (Sections 5 and 6). The first one of these makes direct use of the mechanistic interpretation and concerns a field rotating around an axis (electron, positron?). An analytic expression for a quantity is derived that is highly similar to the so-called fine structure constant. This quantity is not only dimensionless but its determination

can be reduced to finding the solution of a mathematical problem that does not involve any physical parameter, thus to finding a pure mathematical number. Evaluating it, however, requires three nonlinear partial differential equations and two subsequent integrals to be properly solved, which altogether amounts to a task that has so far resisted attempts to finding an analytic solution. Standard numerical methods may have difficulty coping with the specific nature of some of the additional conditions that are to be respected.

The other localized field (photon?) is one that is propagating with the speed of light, c, in a single direction, is planar and transversal with respect to that direction, is polarized either in a fixed direction or circularly, has zero rest energy and zero rest mass, occupies a narrow frequency band centred at its nominal oscillating frequency, and properly obeys the uncertainty relation; its energy is proportional to its nominal frequency, its length proportional to the corresponding wavelength, its momentum equal to its energy divided by c, and in dispersive media it propagates with the group velocity. Such a field obviously is simultaneously wave and particle and thus offers a completely natural explanation of what is known as the wave-particle dualism.

Some of the results presented in this paper have already appeared in [19] and partly in [13-15], but there are also some deeper reaching changes. In particular, the crucial distinction between the three levels of observation for the flow equations and thus the simultaneous relevance of both the classical and the alternative relativistic dynamics had not been realised. Although the relations on which the present viewpoint is based had been included in [19], their far reaching consequences had been overlooked. Instead, an attempt had been made to reconcile the alternative relativistic dynamics with experimental observations, but this way of handling the issue is no longer accepted as adequate. This justifies including in the present paper those earlier results that appear essential for a coherent, self-contained presentation. A summary of the required alternative relativistic results is presented in Appendix C1; details about their derivation can be found in [13-15]. Some concluding and summarizing remarks are given in Section 8. Subsections 3.4 to 3.6, although of definite interest, could be omitted in a first reading.

The author is aware of the many imperfections of this text. Despite its length, it is still quite incomplete, and many details should have been

presented more carefully. In view of his age (83), however, the author feels that the results should be made available at least in their present form. He would be delighted if some younger experts would pick up the thread and address some of the many challenging issues that remain to be examined and clarified.

2. Field velocity and rest field: concepts and basic properties

2.1 Electromagnetic fields in vacuum

Let RF be the *reference frame* under consideration. A point P in RF is characterized by its *position* coordinates x, y, z, or compactly by

$$\mathbf{r} = (x, y, z)^{\mathrm{T}},$$

and its *time* coordinate t, altogether thus by its *coordinates* x, y, z, t. In this text we exclusively consider *electromagnetic* (EM) *fields* in vacuum. Such fields can be described in RF by the *field variables*

$$\mathbf{E} = (E_x, E_y, E_z)^{\mathrm{T}}, \quad \mathbf{H} = (H_x, H_y, H_z)^{\mathrm{T}},$$
$$\mathbf{i} = (i_x, i_y, i_z)^{\mathrm{T}}, \quad \text{and} \quad q,$$

and by two constants, ε and μ, which, obviously, are identical to those frequently denoted by ε_0 and μ_0, respectively. While $\mathbf{E}$, $\mathbf{H}$, $\mathbf{i}$, ε, and μ represent quantities in standard notation (although we here adopt $\mathbf{H}$ instead of the frequently preferred $\mathbf{B}$), we are designating the *charge density* by q, not by ρ as is commonly done. This allows us to represent consistently the density, whether per unit area or per unit volume, of any relevant quantity by a meaningful small letter, and the corresponding full quantity for, say, a particle by the respective capital letter. We also make systematic use of the transposition operator T when handling vectors and matrices; this has several advantages in our context (cf. Appendix A1). We do of course assume all coordinate systems to be right-handed.

The EM field itself is described by what is commonly called *Maxwell's equations*, i.e., by

$$\left.\begin{aligned} \varepsilon\frac{\partial \mathbf{E}}{\partial t}+\mathbf{i}=\nabla\times\mathbf{H}, \quad &\text{(a)}\\ \mu\frac{\partial \mathbf{H}}{\partial t}=-\nabla\times\mathbf{E}, \quad &\text{(b)}\end{aligned}\right\} \tag{2.1}$$

$$q=\varepsilon\nabla^{\mathrm{T}}\mathbf{E}, \quad \nabla^{\mathrm{T}}\mathbf{H}=0, \quad \text{(a),(b)}\Big\} \tag{2.2}$$

where $$\nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)^{\mathrm{T}}.$$

From the point of view adopted in this paper, we assume the field to be *autonomous* (*self-sustaining*), i.e., q and $\mathbf{i}$ to be not sources, but properties of the field. Charges and currents are thus always assumed to be distributed; their densities can be determined from $\mathbf{E}$ and $\mathbf{H}$ by (2.1)(a) and (2.2)(a).

In accordance with standard concepts from special relativity [1-12], let RF' be a second reference frame moving with constant velocity

$$\mathbf{v}_0=(v_{0x},v_{0y},v_{0z})^{\mathrm{T}}$$

with respect to RF. Instead of $\mathbf{v}_0$ we frequently use the normalized velocity $\boldsymbol{\beta}_0$ and the corresponding scalars α_0 and β_0 defined by

$$\boldsymbol{\beta}_0=(\beta_{0x},\beta_{0y},\beta_{0z})^{\mathrm{T}}=\frac{1}{c}\mathbf{v}_0, \quad \beta_0^2=\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\beta}_0, \quad c=\frac{1}{\sqrt{\varepsilon\mu}},$$

$$0\le\beta_0^2<1, \quad \alpha_0=\sqrt{1-\beta_0^2}>0,$$

c being the *speed of light*. As shown explicitly in the definition of α_0, square roots of positive numbers are always assumed positive in this text. To P corresponds in RF' a point P' having coordinates x',y',z',t'. As is commonly done, we make use of the available freedom by adopting for the coordinates of RF and RF' the *privileged choice* for which

$$\mathbf{v}_0=(v_0,0,0)^{\mathrm{T}}, \quad \boldsymbol{\beta}_0=\frac{1}{c}\mathbf{v}_0=(\beta_0,0,0)^{\mathrm{T}}, \quad \beta_0=\frac{v_0}{c}, \tag{2.3}$$

and for which the *Lorentz transformation* linking P and P' thus simplifies to

$$\left.\begin{aligned} x' &= \frac{1}{\alpha_0}(x - v_0 t), \quad y' = y, \quad z' = z, \quad t' = \frac{1}{\alpha_0}(t - \beta_0 \frac{x}{c}), \quad \text{(a)} \\ x &= \frac{1}{\alpha_0}(x' + v_0 t'), \quad y = y', \quad z = z', \quad t = \frac{1}{\alpha_0}(t' + \beta_0 \frac{x'}{c}), \quad \text{(b)} \end{aligned}\right\} \tag{2.4}$$

Since in (2.4) v_0 can be positive or negative, the same holds for β_0, which is thus related to the modulus of $\boldsymbol{\beta}_0$ by $\beta_0 = \pm|\boldsymbol{\beta}_0| = \pm|\beta_0|$. Although the assumption $\beta_0^2 < 1$ excludes the case $\alpha_0 = 0$ we sometimes consider also the limit $\beta_0^2 \to 1$, $\alpha_0 \to 0$.

The field variables in RF′ are

$$\mathbf{E}' = (E_x', E_y', E_z')^{\mathrm{T}}, \quad \mathbf{H}' = (H_x', H_y', H_z')^{\mathrm{T}}, \quad \mathbf{i}' = (i_x', i_y', i_z')^{\mathrm{T}}, \quad q',$$

while ε and μ are, of course, the same as in RF. As is again well-known, it follows from (2.3) and (2.4) that

$$\left.\begin{aligned} E_x' &= E_x, \quad H_x' = H_x, \quad \text{(a),(b)} \\ E_y' &= \frac{1}{\alpha_0}(E_y - \mu v_0 H_z), \quad E_z' = \frac{1}{\alpha_0}(E_z + \mu v_0 H_y), \quad \text{(a),(b)} \\ H_y' &= \frac{1}{\alpha_0}(H_y + \varepsilon v_0 E_z), \quad H_z' = \frac{1}{\alpha_0}(H_z - \varepsilon v_0 E_y), \quad \text{(c),(d)} \end{aligned}\right\} \tag{2.5, 2.6}$$

$$\left.\begin{aligned} i_x' &= \frac{1}{\alpha_0}(i_x - v_0 q), \quad i_y' = i_y, \quad i_z' = i_z, \quad \text{(a),(b),(c)} \\ q' &= \frac{1}{\alpha_0}(q - \frac{v_0}{c^2} i_x), \quad \text{(d)} \end{aligned}\right\} \tag{2.7}$$

the latter equations being of the same general type as (2.4).

The *Poynting vectors* in RF and RF′ are given by

$$\mathbf{S} = (S_x, S_y, S_z)^{\mathrm{T}} = \mathbf{E} \times \mathbf{H}, \qquad \mathbf{S}' = (S_x', S_y', S_z')^{\mathrm{T}} = \mathbf{E}' \times \mathbf{H}',$$

respectively. Using (2.3), (2.5), and (2.6) one finds,

$$\left.\begin{aligned} S'_x &= E'_y H'_z - E'_z H'_y = \frac{1+\beta_0^2}{\alpha_0^2} S_x + \frac{v_0}{\alpha_0^2}(\varepsilon E_x^2 + \mu H_x^2 - 2w), \quad \text{(a)} \\ S'_y &= E'_z H'_x - E'_x H'_z = \frac{1}{\alpha_0}\left(S_y + v_0(\varepsilon E_x E_y + \mu H_x H_y)\right), \quad \text{(b)} \\ S'_z &= E'_x H'_y - E'_y H'_x = \frac{1}{\alpha_0}\left(S_z + v_0(\varepsilon E_x E_z + \mu H_x H_z)\right), \quad \text{(c)} \end{aligned}\right\} \quad (2.8)$$

where

$$S_x = E_y H_z - E_z H_y, \quad S_y = E_z H_x - E_x H_z, \quad S_z = E_x H_y - E_y H_x,$$

and where w is, the classical *field-energy density* in RF given by

$$w = \tfrac{1}{2}(\varepsilon E^2 + \mu H^2), \quad E^2 = \mathbf{E}^{\mathrm{T}}\mathbf{E}, \quad H^2 = \mathbf{H}^{\mathrm{T}}\mathbf{H}. \quad (2.9)$$

For fully fixing E and H, we could in fact assume $E = |\mathbf{E}|$ and $H = |\mathbf{H}|$, thus $E \geq 0$ and $H \geq 0$, but in order to preserve our freedom for later purposes we will not impose that unnecessary restriction. This corresponds to a remark we have made above for v_0 and β_0 and holds for most other vector quantities throughout this paper. Equations such as (2.5) to (2.8) can be inverted by simply interchanging primed and unprimed quantities and by replacing v_0 by $-v_0$ (cf. (2.4)).

2.2 Field velocity

As will be confirmed later, it is crucial to associate with an EM field a further local property that we call the *field velocity*. We represent it by the symbol $\mathbf{v}$ and its associated normalized velocity by $\boldsymbol{\beta}$, and we have,

$$\mathbf{v} = (v_x, v_y, v_z)^{\mathrm{T}}, \quad \boldsymbol{\beta} = (\beta_x, \beta_y, \beta_z)^{\mathrm{T}} = \frac{\mathbf{v}}{c}.$$

In order to arrive at a proper definition of $\mathbf{v}$, recall first that in the usual perception some type of flow is associated with the concept of a Poynting vector $\mathbf{S}$. Hence, it is natural to require $\mathbf{v}$ to be *co-parallel* (cf. first paragraph of Appendix A1) with $\mathbf{S}$ and to vanish if $\mathbf{S} = \mathbf{0}$.

Let us then select any arbitrary point P in RF. The goal we are attempting to achieve is to find an associated reference frame RF′ such that the velocity $\mathbf{v}_0$ by which RF′ moves uniformly with respect to RF has the following properties:

1. $\mathbf{v}_0$ is co-parallel with $\mathbf{S}$ at P in RF.
2. $\mathbf{S}' = \mathbf{0}$ at the corresponding point P′ in RF′.

3. $|\mathbf{v}_0| < c$, thus $|\boldsymbol{\beta}_0| < 1$.

If the three properties just listed lead to a unique result we *define* the *field velocity* $\mathbf{v}$ at P to be equal to the value of $\mathbf{v}_0$ that has been obtained. However, if $w = 0$ we not only have $\mathbf{S} = \mathbf{0}$ but also $\mathbf{S}' = \mathbf{0}$, and this for any choice of $\mathbf{v}_0$ (cf. (2.8)). Therefore, in order to make the problem we are addressing meaningful we assume $w \neq 0$. Or else, we are considering at present only such P for which at least one of the vectors $\mathbf{E}$ and $\mathbf{H}$ is not vanishing. The extension to $w = 0$ will be discussed later.

In order to carry out the calculations now required we first observe that we are still free to choose the position of the x-axis of RF. Hence, we may fix that position by requiring $\mathbf{v}_0$ to be indeed of the form (2.3). But since $\mathbf{S}$ is perpendicular to $\mathbf{E}$ and $\mathbf{H}$, we then have $E_x = H_x = S_y = S_z = 0$. Taking into account (2.8), the requirements $\mathbf{v} = \mathbf{v}_0$ and $\mathbf{S}' = \mathbf{0}$ yield

$$\frac{v_x}{1+\beta^2} = \frac{S_x}{2w}, \quad v_y = v_z = 0, \tag{2.10}$$

or written in vector form,

$$\left.\begin{aligned} &\frac{\mathbf{v}}{1+\beta^2} = \frac{\mathbf{S}}{2w}, \quad \frac{\boldsymbol{\beta}}{1+\beta^2} = \frac{\mathbf{S}}{2cw}, && \text{(a)} \\ &\beta^2 = \boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\beta} = |\boldsymbol{\beta}|^2 = \frac{v^2}{c^2}, \quad v^2 = \mathbf{v}^{\mathrm{T}}\mathbf{v}. && \text{(b)} \end{aligned}\right\} \tag{2.11}$$

Obviously, contrary to (2.10), (2.11) holds for any orientation of $\mathbf{S}$ in RF. It reduces indeed to (2.10) in the special case that had led to that equation and remains valid if we pre-multiply all relevant vectors by a same arbitrary right-handed unit matrix.

Clearly, (2.11) fixes $\boldsymbol{\beta}$ uniquely except for the value of $|\beta|$, for which

$$\frac{2|\beta|}{1+\beta^2} = \frac{|\mathbf{S}|}{cw} \tag{2.12}$$

is obtained. For the Poynting vector we have, using E and H as in (2.9)

$$|\mathbf{S}|^2 = \mathbf{S}^{\mathrm{T}}\mathbf{S} = (\mathbf{E}\times\mathbf{H})^{\mathrm{T}}(\mathbf{E}\times\mathbf{H}) = E^2H^2 - (\mathbf{E}^{\mathrm{T}}\mathbf{H})^2 \tag{2.13}$$

(cf. (A.4), Appendix), and therefore, in view of (2.9) and (2.12),

$$\frac{2|\beta|}{1+\beta^2} \leq \frac{2\sqrt{\varepsilon\mu}\,|EH|}{\varepsilon E^2 + \mu H^2} \leq 1. \tag{2.14}$$

Consequently, the quadratic equation (2.12) always yields positive solutions for $|\beta|$ and thus for β, as needed. Due to (2.13), the first inequality in (2.14) is in fact an equality if and only if $\mathbf{E}^{\mathrm{T}}\mathbf{H} = 0$, and the second one is an equality if and only if $\varepsilon E^2 = \mu H^2$. If both these conditions hold, the solution of (2.12) is $\beta = \pm 1$, thus $v = \pm c$. This limiting solution, which so far has been excluded by the assumption $\alpha_0 > 0$, will henceforth be included in our considerations whenever permitted, even if this is not explicitly stated.

The equation (2.12) remains unchanged if β is replaced by $1/\beta$. Hence, a value with $|\beta| \geq 1$ is always paired with one for which $|\beta| \leq 1$. Only the latter is compatible with the last one (and its limiting case) of the three properties listed above as goal to achieve. The second of those properties is satisfied by construction, and the first one because the scalar factor linking $\mathbf{v}$ and $\mathbf{S}$ in (2.11)(a) is positive. This completes the search for a physically meaningful definition of $\mathbf{v}$.

We may now replace that physics-based definition by a strictly mathematical one. For doing this, we define the field velocity $\mathbf{v}$ of an EM field in a point P by means of (2.11) together with the requirement $|\beta| \leq 1$. Furthermore, we denote by RF_0 the specific RF′ we have selected and call it the *rest reference frame* associated with P. Thus, the uniform velocity by which RF_0 moves with respect to RF is equal to the field velocity $\mathbf{v}$ at the point P with which RF_0 is associated. Due to $\mathbf{S}^{\mathrm{T}}\mathbf{E} = \mathbf{S}^{\mathrm{T}}\mathbf{H} = 0$ and (2.11) we have (cf. (A.3)),

$$\mathbf{v}^{\mathrm{T}}\mathbf{E} = \mathbf{v}^{\mathrm{T}}\mathbf{H} = 0, \quad \boldsymbol{\beta}^{\mathrm{T}}\mathbf{E} = \boldsymbol{\beta}^{\mathrm{T}}\mathbf{H} = 0. \tag{2.15}$$

Since quantities such as β, $\boldsymbol{\beta}$, β_0, $\boldsymbol{\beta}_0$, etc. differ from the corresponding v, $\mathbf{v}$, v_0, $\mathbf{v}_0$ etc. only by the normalizing factor $1/c$, we sometimes refer to the former indifferently as *velocities* in the same way as to the latter. If clarity requires, we may refer to the former as *normalized* velocities, to the latter as *actual* velocities.

2.3 Basic properties of the rest field

In line with the results of Subsection 2.2, the electric and magnetic fields that can be associated with P by way of RF_0 and have so far been de-

noted by $\mathbf{E}'$ and $\mathbf{H}'$ will be called the *electric rest field* and the *magnetic rest field* at P. We denote them henceforth by

$$\mathbf{E}_0 = (E_{0x}, E_{0y}, E_{0z})^{\mathrm{T}}, \quad \mathbf{H}_0 = (H_{0x}, H_{0y,} H_{0z})^{\mathrm{T}}.$$

Hence, adopting for the coordinate axes the privileged choice (2.3), we obtain from (2.5) and(2.6),

$$\left.\begin{aligned} &E_{0x} = 0, \quad E_{0y} = \frac{1}{\alpha}(E_y - \mu v H_z), \quad E_{0z} = \frac{1}{\alpha}(E_z + \mu v H_y), \qquad \text{(a)} \\ &H_{0x} = 0, \quad H_{0y} = \frac{1}{\alpha}(H_y + \varepsilon v E_z), \quad H_{0z} = \frac{1}{\alpha}(H_z - \varepsilon v E_y), \quad \text{(b)} \end{aligned}\right\} \tag{2.16}$$

where

$$\alpha = \sqrt{1-\beta^2}, \tag{2.17}$$

v and β being as defined before (cf. (2.11)). As can be verified, the result may also be written in the form

$$\left.\mathbf{E}_0 = \frac{1}{\alpha}(\mathbf{E} + \mu \mathbf{v} \times \mathbf{H}), \quad \mathbf{H}_0 = \frac{1}{\alpha}(\mathbf{H} - \varepsilon \mathbf{v} \times \mathbf{E}), \quad \text{(a),(b)}\right\} \tag{2.18}$$

which holds for any orientation of $\mathbf{v}$, thus in all generality. As we had argued for (2.11), (2.18) reduces indeed to (2.16) if $\mathbf{v} = (v, 0, 0)^{\mathrm{T}}$ and remains valid if $\mathbf{E}$, $\mathbf{H}$, $\mathbf{E}_0$, $\mathbf{H}_0$, and $\mathbf{v}$ are replaced by $\mathbf{UE}$, $\mathbf{UH}$, $\mathbf{UE}_0$, $\mathbf{UH}_0$, and $\mathbf{Uv}$, respectively, where $\mathbf{U}$ is an arbitrary right-handed unit matrix. Clearly, we may henceforth ignore the reference frame RF_0 and interpret $\mathbf{E}_0$ and $\mathbf{H}_0$ simply as vectors associated with the point P in RF.

From (2.18) we also derive, using (2.15) and (A.3),

$$\mathbf{v}^{\mathrm{T}}\mathbf{E}_0 = 0, \qquad \mathbf{v}^{\mathrm{T}}\mathbf{H}_0 = 0. \tag{2.19}$$

Hence, $\mathbf{v}$ is also orthogonal to $\mathbf{E}_0$ and $\mathbf{H}_0$. Since

$$\left.\mathbf{S}_0 = \mathbf{0}, \quad \mathbf{S}_0 := \mathbf{E}_0 \times \mathbf{H}_0, \quad \text{(a),(b)}\right\} \tag{2.20}$$

i.e., since $\mathbf{E}_0$ and $\mathbf{H}_0$ are parallel, it is useful to write them, equivalently, in the form

$$\left.\begin{aligned} &\mathbf{E}_0 = \boldsymbol{\gamma} E_0, \quad \mathbf{H}_0 = \boldsymbol{\gamma} H_0, \qquad \text{(a)} \\ &\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\gamma} = 1, \quad \boldsymbol{\gamma} = (\gamma_x, \gamma_y, \gamma_z)^{\mathrm{T}}, \quad \text{(b)} \end{aligned}\right\} \tag{2.21}$$

$\boldsymbol{\gamma}$ thus being an appropriate unit vector. In view of (2.21)(a), the signs of the scalars E_0 and H_0 may not be chosen independently, but there re-

mains some freedom since multiplying E_0, H_0, and $\boldsymbol{\gamma}$ by -1 does not affect the validity of (2.21). Due to (2.19),

$$\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\gamma} = \mathbf{v}^{\mathrm{T}}\boldsymbol{\gamma} = 0\,, \tag{2.22}$$

provided at least one of the quantities E_0 and H_0 does not vanish.

We can solve (2.18) for $\mathbf{E}$ and $\mathbf{H}$ by using (A.2) and (2.15) while leaving out other details of $\mathbf{v}$; this yields

$$\left.\mathbf{E} = \frac{1}{\alpha}(\mathbf{E}_0 - \mu\mathbf{v}\times\mathbf{H}_0), \quad \mathbf{H} = \frac{1}{\alpha}(\mathbf{H}_0 + \varepsilon\mathbf{v}\times\mathbf{E}_0). \quad \text{(a),(b)}\right\} \tag{2.23}$$

Similarly, we can solve (2.23) for $\mathbf{E}_0$ and $\mathbf{H}_0$ by using (A.2) and (2.19) while leaving out other details of $\mathbf{v}$; this yields (2.18). Clearly, whether we consider (2.18) or (2.23), (2.15) always implies (2.19) and vice versa.

It appears natural to associate with P a *rest energy density*

$$w_0 = \tfrac{1}{2}(\varepsilon E_0^2 + \mu H_0^2). \tag{2.24}$$

Using (2.21) to (2.23), we obtain (cf. (A.3) and (A.4)),

$$\left.\begin{aligned} \alpha^2\varepsilon E^2 &= \varepsilon E_0^2 - \frac{2}{c^2}\mathbf{E}_0^{\mathrm{T}}(\mathbf{v}\times\mathbf{H}_0) + \mu\,|\,\boldsymbol{\beta}\times\mathbf{H}_0\,|^2 \\ &= \varepsilon E_0^2 + \beta^2\mu H_0^2 \qquad \text{(a)} \\ \alpha^2\mu H^2 &= \mu H_0^2 + \frac{2}{c^2}\mathbf{H}_0^{\mathrm{T}}(\mathbf{v}\times\mathbf{E}_0) + \varepsilon\,|\,\boldsymbol{\beta}\times\mathbf{E}_0\,|^2 \\ &= \mu H_0^2 + \beta^2\varepsilon E_0^2 \qquad \text{(b)} \end{aligned}\right\} \tag{2.25}$$

and thus

$$\alpha^2 w = (1-\beta^2)w = (1+\beta^2)w_0\,, \tag{2.26}$$

where w is as defined by (2.9). Furthermore, we deduce first from (2.25) and then from (2.23) and (A.4),

$$\left.\varepsilon E^2 - \mu H^2 = \varepsilon E_0^2 - \mu H_0^2, \quad \mathbf{E}^{\mathrm{T}}\mathbf{H} = \mathbf{E}_0^{\mathrm{T}}\mathbf{H}_0 = E_0 H_0\,, \quad \text{(a),(b)}\right\} \tag{2.27}$$

respectively, and thus also, taking into account (2.13) and (2.24),

$$\left.\begin{aligned} w_0^2 &= \frac{1}{4}(\varepsilon E_0^2 - \mu H_0^2)^2 + \frac{1}{c^2}(E_0 H_0)^2\,, \quad \text{(a)} \\ &= \frac{1}{4}(\varepsilon E^2 - \mu H^2)^2 + \frac{1}{c^2}(\mathbf{E}^{\mathrm{T}}\mathbf{H})^2\,, \quad \text{(b)} \\ &= w^2 - \frac{1}{c^2}\,|\,\mathbf{S}\,|^2\,. \qquad \text{(c)} \end{aligned}\right\} \tag{2.28}$$

Using (2.26) we derive from (2.11),

$$\mathbf{S} = 2\frac{w_0}{\alpha^2}\mathbf{v} = 2c\frac{w_0}{\alpha^2}\boldsymbol{\beta}\,. \tag{2.29}$$

So far we have been guided in our derivations by physical arguments. Instead, we could take a purely mathematical point of view and attempt to define $\mathbf{E}_0$ and $\mathbf{H}_0$ by means of (2.17) and (2.18), with $\mathbf{v}$ required to be such that (2.15) and (2.20) hold. For doing this, we first substitute (2.18) in (2.20)(b). Using (2.9), (A.2), and (A.5), this yields,

$$\left.\begin{aligned}\alpha^2\mathbf{S}_0 &= \mathbf{E}\times\mathbf{H} - \varepsilon\mathbf{E}\times(\mathbf{v}\times\mathbf{E}) - \mu\mathbf{H}\times(\mathbf{v}\times\mathbf{H}) + \varepsilon\mu(\mathbf{v}\times\mathbf{E})\times(\mathbf{v}\times\mathbf{H})\\ &= \mathbf{S} + (\boldsymbol{\beta}^{\mathrm{T}}\mathbf{S})\boldsymbol{\beta} - 2wc\boldsymbol{\beta} + \varepsilon(\mathbf{v}^{\mathrm{T}}\mathbf{E})\mathbf{E} + \mu(\mathbf{v}^{\mathrm{T}}\mathbf{H})\mathbf{H},\end{aligned}\right\} \tag{2.30}$$

and therefore, due to (2.20)(a) and (2.15),

$$\mathbf{S} + (\boldsymbol{\beta}^{\mathrm{T}}\mathbf{S} - 2wc)\boldsymbol{\beta} = 0\,.$$

Hence, $\boldsymbol{\beta}$ has to be parallel to $\mathbf{S}$. This in turn implies $(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{S})\boldsymbol{\beta} = \beta^2\mathbf{S}$ and therefore indeed (2.11). All other results (2.19) to (2.29) then follow as above. Vice versa, if $\boldsymbol{\beta}$, $\mathbf{E}_0$, and $\mathbf{H}_0$ are defined by (2.11), (2.17), and (2.18), respectively, (2.15) and (2.20) can be verified to hold and thus all other results subsequent to (2.18). Consequently, the desired solution exists for any choice of $\mathbf{E}$ and $\mathbf{H}$, thus for all 6 degrees of freedom that correspond to the 6 components of these two vectors, and is unique.

Alternatively, we may start by choosing any $\mathbf{E}_0$, $\mathbf{H}_0$, and $\mathbf{v}$ that satisfy (2.21) and (2.22). This also involves 6 degrees of freedom, i.e., E_0, H_0, the 3 components of $\mathbf{v}$, and the one degree of freedom that is left for the unit vector $\boldsymbol{\gamma}$ after requiring it to be orthogonal to $\mathbf{v}$ (cf. (2.22)). We then introduce $\mathbf{E}$ and $\mathbf{H}$ by (2.23), which immediately yields, correspondingly to (2.30),

$$\begin{aligned}\alpha^2\mathbf{S} &= \alpha^2\mathbf{E}\times\mathbf{H}\\ &= \mathbf{S}_0 + 2w_0\mathbf{v} - \varepsilon(\mathbf{v}^{\mathrm{T}}\mathbf{E}_0)\mathbf{E}_0 - \mu(\mathbf{v}^{\mathrm{T}}\mathbf{H}_0)\mathbf{H}_0 + \boldsymbol{\beta}\boldsymbol{\beta}^{\mathrm{T}}\mathbf{S}_0.\end{aligned}$$

This in turn reduces to (2.29) if we take into account (2.21) (which implies (2.20)) and (2.22). All remaining results can then be derived in a way similar to that used above. Unless otherwise stated, we will however formulate our subsequent discussions in terms of the original alternative.

2.4 Further properties of the rest field

We can determine β^2 easiest by means of (2.26), i.e. by

$$\beta^2 = \frac{w - w_0}{w + w_0}, \tag{2.31}$$

where, due to (2.9) and (2.28), w and w_0 are both known. As follows from (2.28) and (2.31), w_0 can cover the range

$$0 \le w_0 \le w, \tag{2.32}$$

where both limits can indeed be reached. Accordingly, (2.31) yields values for which

$$0 \le |\beta| \le 1, \quad 0 \le |v| \le c \tag{2.33}$$

Clearly, (2.31) would yield a value $|\beta| > 1$ (cf. the discussion subsequent to (2.14)) for $w_0 < 0$, but this would not be compatible with the defining equation (2.24).

At a given point P, the field velocity can take any of the values allowed by (2.33). More specifically, if it reaches the lower bound $\mathbf{v} = \mathbf{0}$, to which corresponds $\alpha = 1$, we say that the field is *locally at rest* at P. According to (2.11), this occurs if and only if $\mathbf{S} = \mathbf{0}$, and according to (2.31), if and only if $w = w_0$. Correspondingly, we may say that a field is *fully at rest* if it is locally at rest everywhere. This is in particular the case if the field is either purely *electrostatic* or purely *magnetostatic*.

The other extreme, the upper bound $|v| = c$, for which $\alpha = 0$, is reached at P if and only if $w_0 = 0$, thus, according to (2.28), if and only if

$$\varepsilon E^2 = \mu H^2, \quad \mathbf{E}^{\mathrm{T}}\mathbf{H} = 0 \ , \tag{2.34}$$

or else, if and only if the field is locally planar (behaves at P like a planar wave); this agrees with what could be expected in view of classical photon concepts (cf. Section 6). As follows from (2.24), $w_0 = 0$ also implies $\mathbf{E}_0 = \mathbf{H}_0 = \mathbf{0}$.

In view of (2.29) and (A.2) we also have

$$\left.\begin{aligned} \frac{2w_0}{\alpha^2}\mathbf{v}\times\mathbf{H} &= (\mathbf{E}\times\mathbf{H})\times\mathbf{H} = (\mathbf{E}^{\mathrm{T}}\mathbf{H})\mathbf{H} - H^2\mathbf{E}, \quad \text{(a)} \\ \frac{2w_0}{\alpha^2}\mathbf{v}\times\mathbf{E} &= (\mathbf{E}\times\mathbf{H})\times\mathbf{E} = E^2\mathbf{H} - (\mathbf{E}^{\mathrm{T}}\mathbf{H})\mathbf{E}, \quad \text{(b)} \end{aligned}\right\} \tag{2.35}$$

where E^2 and H^2 can be expressed by means of (2.25). Evaluating then the right-hand sides of (2.18) by means of (2.35) and taking into account (2.21), (2.25), and (2.27)(b) one finds

$$\frac{2w_0}{\alpha}\mathbf{E}_0 = \frac{2w_0}{\alpha}E_0\boldsymbol{\gamma} = \varepsilon E_0^2\mathbf{E} + \mu E_0 H_0\mathbf{H},$$
$$\frac{2w_0}{\alpha}\mathbf{H}_0 = \frac{2w_0}{\alpha}H_0\boldsymbol{\gamma} = \varepsilon E_0 H_0\mathbf{E} + \mu H_0^2\mathbf{H}.$$

Provided E_0 and H_0 are not vanishing simultaneously, both equations yield for $\boldsymbol{\gamma}$ the interesting expression

$$\boldsymbol{\gamma} = \frac{\alpha}{2w_0}(\varepsilon E_0\mathbf{E} + \mu H_0\mathbf{H}), \tag{2.36}$$

which can be verified to satisfy indeed $\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\gamma} = 1$. According to (2.21), the direction of $\boldsymbol{\gamma}$ is undefined if $E_0 = H_0 = 0$, thus if $\mathbf{E} = \mathbf{H} = \mathbf{0}$.

2.5 Rest charge, rest current, and charge velocity

For the *rest current density* (per unit cross area)

$$\mathbf{i}_0 = (i_{0x}, i_{0y}, i_{0z})^{\mathrm{T}}$$

and the *rest charge density* q_0 (per unit volume) we deduce from (2.7),

$$\left.\begin{aligned} \mathbf{i}_0 &= \mathbf{i} - \frac{1}{\alpha}\left(q - \frac{\mathbf{v}^{\mathrm{T}}\mathbf{i}}{(1+\alpha)c^2}\right)\mathbf{v} = \mathbf{i} - \frac{1}{\alpha}\left(cq - \frac{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{i}}{1+\alpha}\right)\boldsymbol{\beta}, \quad \text{(a)} \\ q_0 &= \frac{1}{\alpha}\left(q - \frac{1}{c^2}\mathbf{v}^{\mathrm{T}}\mathbf{i}\right) = \frac{1}{\alpha}\left(q - \frac{1}{c}\boldsymbol{\beta}^{\mathrm{T}}\mathbf{i}\right), \quad \text{(b)} \end{aligned}\right\} \tag{2.37}$$

which we have directly written in the general form as had been done for $\mathbf{E}_0$ and $\mathbf{H}_0$ in (2.18). For justifying (2.37)(a), it is convenient to first write the result deduced from (2.7) in the form

$$\mathbf{i}_0 = \begin{pmatrix} i_x/\alpha \\ i_y \\ i_z \end{pmatrix} - \frac{q}{\alpha}\begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} = \mathbf{i} + \frac{1}{\alpha}((1-\alpha)i_x - vq)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{2.38}$$

We then observe that $1-\alpha=\beta^2/(1+\alpha)$ and that in (2.38) we have $\mathbf{v}=(v,0,0)^{\mathrm{T}}$ and therefore $vi_x=\mathbf{v}^{\mathrm{T}}\mathbf{i}$.

Inverting (2.37) leads to

$$\left.\begin{aligned}\mathbf{i}&=\mathbf{i}_0+\frac{1}{\alpha}\left(q_0+\frac{\mathbf{v}^{\mathrm{T}}\mathbf{i}_0}{(1+\alpha)c^2}\right)\mathbf{v} \quad \text{(a)}\\ &=\mathbf{i}_0+q\mathbf{v}-\frac{1}{1+\alpha}(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{i}_0)\boldsymbol{\beta}, \quad \text{(b)}\\ q&=\frac{1}{\alpha}(q_0+\frac{1}{c^2}\mathbf{v}^{\mathrm{T}}\mathbf{i}_0), \quad \text{(c)}\end{aligned}\right\} \tag{2.39}$$

where (2.39)(b) is obtained by eliminating q_0 between (2.39)(a) and (c). Clearly, as should be the case, (2.37) and (2.39) can simply be deduced from one another by interchanging q and q_0, $\mathbf{i}$ and $\mathbf{i}_0$, and by replacing $\mathbf{v}$ by $-\mathbf{v}$, thus in a way similar to what we can observe for (2.18) and (2.23).

Some further useful relations can be derived from the above. Thus, elimination of either $\mathbf{v}^{\mathrm{T}}\mathbf{i}_0$ or q_0 between (2.39)(a) and (b) yields first

$$\left.\begin{aligned}\mathbf{i}&=\mathbf{i}_0+\frac{q+q_0}{1+\alpha}\mathbf{v} \quad \text{(a)}\\ &=q\mathbf{v}+\mathbf{i}_0-\frac{1}{1+\alpha}\boldsymbol{\beta}\boldsymbol{\beta}^{\mathrm{T}}\mathbf{i}_0, \quad \text{(b)}\end{aligned}\right\} \tag{2.40}$$

respectively. If we then pre-multiply (2.40)(a) first by $\mathbf{E}^{\mathrm{T}}$ and then by $\mathbf{E}_0^{\mathrm{T}}$ we obtain, due to (2.15) and (2.19),

$$\mathbf{i}^{\mathrm{T}}\mathbf{E}=\mathbf{i}_0^{\mathrm{T}}\mathbf{E}, \quad \mathbf{i}^{\mathrm{T}}\mathbf{E}_0=\mathbf{i}_0^{\mathrm{T}}\mathbf{E}_0, \tag{2.41}$$

while pre-multiplying (2.40)(b) first by $\mathbf{v}^{\mathrm{T}}$ and then by $\boldsymbol{\gamma}^{\mathrm{T}}$ yields, due to (2.22) and $\beta^2/(1+\alpha)=1-\alpha$,

$$\alpha\mathbf{v}^{\mathrm{T}}\mathbf{i}_0=\mathbf{v}^{\mathrm{T}}\mathbf{i}-v^2q, \quad \boldsymbol{\gamma}^{\mathrm{T}}\mathbf{i}=\boldsymbol{\gamma}^{\mathrm{T}}\mathbf{i}_0 \quad \text{(a),(b)}\Big\}. \tag{2.42}$$

Furthermore, one can derive, for instance from (2.39)(b) and (2.40)(a),

$$c^2q^2-|\mathbf{i}|^2=c^2q_0^2-|\mathbf{i}_0|^2. \tag{2.43}$$

As has been done for $\mathbf{E}_0$ and $\mathbf{H}_0$ (cf. Subsection 2.3), we may also in the present context ignore the reference frame RF_0 and interpret $\mathbf{i}_0$ and q_0 simply as quantities associated with the point P in RF.

Let us still define the *charge velocity* $\mathbf{v}_{ch}$, i.e. the velocity by which a charge density $q \neq 0$ is moving in order to create the current density $\mathbf{i}$, by

$$\mathbf{i} = q\mathbf{v}_{ch} = cq\boldsymbol{\beta}_{ch}, \quad \boldsymbol{\beta}_{ch} = \frac{1}{c}\mathbf{v}_{ch}.$$

We obviously have $\mathbf{v}^{\mathrm{T}}\mathbf{i} = q\mathbf{v}^{\mathrm{T}}\mathbf{v}_{ch}$ and thus from (2.37),

$$q_0 = \frac{q}{\alpha}(1-\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\beta}_{ch}), \quad \mathbf{i}_0 = q\left(\mathbf{v}_{ch} - \frac{1}{\alpha}(1-\frac{1}{(1+\alpha)}\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\beta}_{ch})\mathbf{v}\right).$$

In particular, for $\mathbf{v}_{ch} = \mathbf{v}$, thus for $\mathbf{i} = q\mathbf{v}$, we find

$$q_0 = \alpha q, \qquad \mathbf{i}_0 = \mathbf{0}. \tag{2.44}$$

Vice versa, if $\mathbf{i}_0 = \mathbf{0}$ we deduce from (2.39),

$$q_0 = \alpha q, \qquad \mathbf{i} = q\mathbf{v}, \tag{2.45}$$

thus also $\mathbf{v} = \mathbf{v}_{ch}$, and furthermore $q_0 = 0$ if we also have $|v| = c$. We also conclude from this that $\mathbf{i} = q\mathbf{v}$ is necessary and sufficient for ensuring $\mathbf{i}_0 = \mathbf{0}$. If we accept that $\mathbf{v}_{ch}$ has a true physical meaning, we must always have

$$|\mathbf{v}_{ch}| \leq c, \quad |\boldsymbol{\beta}_{ch}| \leq 1,$$

which, due to (2.43), implies

$$|\mathbf{i}_0|^2 = c^2(q_0^2 - q^2) + |\mathbf{i}|^2 = c^2q_0^2 + q^2(|\mathbf{v}_{ch}|^2 - c^2)$$

and thus is equivalent to $|\mathbf{i}_0| \leq c|q_0|$.

2.6 Classical energy velocity

In view of the interpretation usually given to $\mathbf{S}$ and w, the quantity $\mathbf{v}_c$ defined by

$$\mathbf{S} = \mathbf{v}_c w \tag{2.46}$$

may be called the *classical energy velocity*. Comparing with (2.11)(a) we find,

$$\mathbf{v}_c = \frac{2}{1+\beta^2}\mathbf{v}. \tag{2.47}$$

In particular, $\mathbf{v}_c$ and $\mathbf{v}$ are co-parallel and

$$|\mathbf{v}_c| \geq |\mathbf{v}|,$$

equality holding only for $\mathbf{v} = 0$ and $|\mathbf{v}| = c$. For small values of β we have $\mathbf{v}_c = 2\mathbf{v}$, thus nowhere $\mathbf{v}_c = \mathbf{v}$ as one might superficially expect, and the ratio $|\mathbf{v}_c|/|\mathbf{v}|$ decreases monotonically for increasing values of $|\mathbf{v}|$.

Defining

$$\boldsymbol{\beta}_c = \frac{1}{c}\mathbf{v}_c, \quad \boldsymbol{\beta}_c^{\mathrm{T}}\boldsymbol{\beta}_c = \beta_c^2, \quad \alpha_c^2 = 1 - \beta_c^2,$$

we derive from (2.47),

$$\boldsymbol{\beta}_c = \frac{2}{1+\beta^2}\boldsymbol{\beta}, \quad \alpha_c = \frac{\alpha^2}{1+\beta^2} = \frac{1-\beta^2}{1+\beta^2}, \quad \frac{2\mathbf{v}}{\alpha^2} = \frac{\mathbf{v}_c}{\alpha_c}. \tag{2.48}$$

Due to (2.26) we thus also have

$$w = \frac{w_0}{\alpha_c}, \tag{2.49}$$

which shows that w_0 is also of interest for w. From (2.48) together with (2.9), (2.12), and (2.13) we obtain

$$\beta_c^2 = \frac{4\beta^2}{(1+\beta^2)^2} = 4\frac{(EH)^2 - (\mathbf{E}^{\mathrm{T}}\mathbf{H})^2}{(\varepsilon E^2 + \mu H^2)^2 c^2}. \tag{2.50}$$

2.7 Further velocity aspects

All velocities encountered so far are defined only in terms of the local values of the original field variables $\mathbf{E}$, $\mathbf{H}$, q, and $\mathbf{i}$, i.e., the values these variables admit at the point P under consideration. In other words, they do not depend on derivatives with respect to the coordinates and thus on the way $\mathbf{E}$, $\mathbf{H}$, q, and $\mathbf{i}$ depend on x, y, z, and t. In fact, the specifics of such dependencies determine some further propagation aspects. Although one would expect that corresponding velocities are essentially equal to the field velocity, one may not simply assume them to be identical to the latter. Furthermore, if a field that is essentially localized in a small volume (particle?) is moving as a whole, the velocity of that movement cannot be expected to be simply equal to some specific field velocity. This issue will be examined further in later sections, in particular in Section 7.

2.8 Lorentz invariances

We consider a Lorentz transformation as discussed at the beginning of Subsection 2.1 (cf. (2.3) to (2.7)). We continue using un-primed symbols when referring to quantities in RF and primed symbols for the corresponding quantities in RF'. The two following Lorentz invariances are known to hold:

$$\varepsilon E'^2 - \mu H'^2 = \varepsilon E^2 - \mu H^2, \quad \mathbf{E}'^{\mathrm{T}}\mathbf{H}' = \mathbf{E}^{\mathrm{T}}\mathbf{H}. \quad \text{(a),(b)}\Big\} \tag{2.51}$$

As (2.28)(b) thus shows, the rest energy density w_0 is *Lorentz invariant*, i.e.,

$$w_0' = \tfrac{1}{2}(\varepsilon E_0'^2 + \mu H_0'^2) = w_0 . \tag{2.52}$$

This is remarkable since the classical energy density w (cf. (2.9) is known *not* to be Lorentz invariant. Furthermore, in view of (2.27)(a) and (2.51)(a), $\varepsilon E_0^2 - \mu H_0^2$ is also Lorentz invariant and so are therefore E_0^2 and H_0^2 individually. Consequently, we can write,

$$E_0'^2 = E_0^2, \quad H_0'^2 = H_0^2 .$$

But according to (2.27)(b) and (2.51)(b) we also have $E_0'H_0' = E_0H_0$. The freedom for choosing the signs of E_0 and H_0 mentioned just before (2.22) exists correspondingly for E_0' and H_0'. We are thus free to require $E_0' = E_0$, as will always be done hereafter, and we then obtain altogether,

$$E_0' = E_0, \quad H_0' = H_0, \quad w_0' = w_0 . \tag{2.53}$$

Another Lorentz invariance exists between $\mathbf{i}$ and q, i.e.,

$$c^2q'^2 - |\mathbf{i}'|^2 = c^2q^2 - |\mathbf{i}|^2 .$$

Hence, in view of (2.43), $c^2q_0^2 - |\mathbf{i}_0|^2$ is also Lorentz invariant, but no such invariance exists for $\mathbf{i}_0$ or q_0 alone.

2.9 Basal electromagnetic field

It appears reasonable to assume that the existence of a non-vanishing current density is always the result of a moving charge density. This justifies introducing the concept of a basal EM field. More precisely, an autonomous EM field will be called *basal* in a given reference frame RF if $\mathbf{i}_0 = \mathbf{0}$ holds everywhere in RF. The judiciousness of such an assumption is confirmed by counting the available number of degrees of freedom.

There are indeed 7 such degrees, 6 of which correspond to E_0, H_0, q_0, and the 3 components of $\mathbf{v}$, the 7th being the degree of freedom remaining in $\boldsymbol{\gamma}$ after taking into account $\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\gamma}=1$ and $\mathbf{v}^{\mathrm{T}}\boldsymbol{\gamma}=0$. As thus needed, there are also exactly 7 individual equations that have to be verified, i.e., the 6 individual equations in (2.1) and the equation (2.2)(a). In that count, the remaining equation (2.2)(b) had to be ignored because it amounts simply to an initial condition; from (2.1)(b) we obtain indeed

$$\frac{\partial\nabla^{\mathrm{T}}\mathbf{H}}{\partial t}=0\,, \tag{2.54}$$

due to which $\nabla^{\mathrm{T}}\mathbf{H}=0$ holds for all t as soon as it does so for an arbitrary single time instant, for instance for the initial time t_0.

If an EM field is basal, (2.39) simplifies to (cf. (2.44) and (2.45)),

$$q_0=\alpha q,\quad \mathbf{i}=q\mathbf{v}.\quad \text{(a),(b)}\Big\} \tag{2.55}$$

Due to this, (2.7) in turn simplifies to

$$\left.\begin{aligned} &i'_x=\frac{1}{\alpha_0}(v_x-v_0)q,\quad i'_y=v_y q,\quad i'_z=v_z q,\\ &q'=\frac{1}{\alpha_0}(1-\beta_0\beta_x)q=\frac{1-\beta_0\beta_x}{\alpha\alpha_0}q_0. \end{aligned}\right\} \tag{2.56}$$

It is instructive to repeat the aforementioned count by starting from the same equations as before but without appealing to the rest field. We may then express $\mathbf{i}$ in terms of $\mathbf{v}$ by means of (2.55)(b), express in turn $\mathbf{v}$ by means of (2.11)(a), and eliminate q by means of (2.2)(a). The complete set of equations of a basal field then becomes

$$\varepsilon\Big(\frac{\partial\mathbf{E}}{\partial t}+\mathbf{v}\nabla^{\mathrm{T}}\mathbf{E}\Big)=\nabla\times\mathbf{H},\quad \mu\frac{\partial\mathbf{H}}{\partial t}=-\nabla\times\mathbf{E},$$
$$\frac{c^2\mathbf{v}}{c^2+\mathbf{v}^{\mathrm{T}}\mathbf{v}}=\frac{\mathbf{E}\times\mathbf{H}}{\varepsilon\mathbf{E}^{\mathrm{T}}\mathbf{E}+\mu\mathbf{H}^{\mathrm{T}}\mathbf{H}}$$

and is obviously nonlinear. It comprises 9 individual equations in the 9 degrees of freedom contained in $\mathbf{E}$, $\mathbf{H}$, and $\mathbf{v}$. In addition, as mentioned in the context of (2.54), $\nabla^{\mathrm{T}}\mathbf{H}=0$ must be taken into account for instance as an initial condition. Alternatively, the above three equations may be replaced by

$$\varepsilon\frac{\partial \mathbf{E}}{\partial t}+\mathbf{v}q=\nabla\times\mathbf{H},\quad \mu\frac{\partial \mathbf{H}}{\partial t}=-\nabla\times\mathbf{E},$$
$$\frac{\partial q}{\partial t}+\nabla^{\mathrm{T}}(q\mathbf{v})=0,\quad \frac{c^2\mathbf{v}}{c^2+\mathbf{v}^{\mathrm{T}}\mathbf{v}}=\frac{\mathbf{E}\times\mathbf{H}}{\varepsilon\mathbf{E}^{\mathrm{T}}\mathbf{E}+\mu\mathbf{H}^{\mathrm{T}}\mathbf{H}}.$$

This set of four equations comprises 10 individual equations and correspondingly involves 10 degrees of freedom, i.e. those contained in $\mathbf{E}$, $\mathbf{H}$, $\mathbf{v}$, and q. The first and third of the four new equations yield indeed consecutively,

$$\frac{\partial(\varepsilon\nabla^{\mathrm{T}}\mathbf{E})}{\partial t}+\nabla^{\mathrm{T}}(q\mathbf{v})=0,\quad \frac{\partial(\varepsilon\nabla^{\mathrm{T}}\mathbf{E}-q)}{\partial t}=0,$$

and are thus obviously compatible with $q=\varepsilon\nabla^{\mathrm{T}}\mathbf{E}$. Or else, if we define q by means of $q=\varepsilon\nabla^{\mathrm{T}}\mathbf{E}$ at one single time instant, say again at the initial time t_0, that same definition is automatically verified for any t, as required.

One may conjecture that an autonomous EM field is always basal in some reference frame. If that should indeed be the case there would be no essential difference between the two classes of fields. This would agree with the intuitive impression that if in an autonomous field the current density is non-vanishing at a certain location, the charge density at that same location can neither be vanishing nor be at rest.

Note that substantial simplifications still occur if instead of $\mathbf{i}_0=\mathbf{0}$ one imposes the less restrictive requirement that $\mathbf{i}_0$ and $\mathbf{v}$ are orthogonal. We then have (cf. (2.39)),

$$\mathbf{v}^{\mathrm{T}}\mathbf{i}_0=0,\quad \text{thus}\quad q_0=\alpha q,\quad \mathbf{i}=\mathbf{i}_0+q\mathbf{v}. \tag{2.57}$$

This is the case in particular if $\mathbf{i}_0=\boldsymbol{\gamma}i_0$, thus for fields that in [19] have been called (possibly somewhat inappropriately) *intrinsic*, or *fully intrinsic* if they are intrinsic everywhere. Such fields have altogether 8 degrees of freedom, i.e., if one adopts the same point of view as in the discussion preceding (2.54) above. In the light of what has been said about (2.54), the statement in [19] that 8 is also the number of individual equations in the original set (2.1) and (2.2), obviously, cannot be used for justifying the need for the additional parameter i_0. The issue of intrinsic fields will not be given major attention in the present text (except, indirectly, in the

sense that they are partly related to basal fields); details can be found in [19].

2.10 Local decomposition into oppositely propagating planar waves

Assume the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ to be non-vanishing at the point P under consideration. One can then always decompose $\mathbf{E}$ and $\mathbf{H}$ according to

$$\left.\begin{aligned} \mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-, \quad \mathbf{H} = \mathbf{H}_+ + \mathbf{H}_-, \quad \text{(a)} \\ q = q_+ + q_-, \quad \mathbf{i} = \mathbf{i}_+ + \mathbf{i}_- \qquad \text{(b)} \end{aligned}\right\} \tag{2.58}$$

into two locally planar fields, thus into fields that satisfy

$$\left.\begin{aligned} \varepsilon \mathbf{E}_+^{\mathrm{T}} \mathbf{E}_+ = \mu \mathbf{H}_+^{\mathrm{T}} \mathbf{H}_+, \quad \varepsilon \mathbf{E}_-^{\mathrm{T}} \mathbf{E}_- = \mu \mathbf{H}_-^{\mathrm{T}} \mathbf{H}_-, \\ \mathbf{E}_+^{\mathrm{T}} \mathbf{H}_+ = \mathbf{E}_-^{\mathrm{T}} \mathbf{H}_- = 0. \end{aligned}\right\} \tag{2.59}$$

The one field (characterized by subscripts +) propagates in the direction of $\mathbf{S}$, the other one (characterized by subscripts -) oppositely to $\mathbf{S}$. This decomposition is unique and its solution for $\mathbf{E}_+$, $\mathbf{E}_-$, $\mathbf{H}_+$, and $\mathbf{H}_-$ is given by (cf. Appendix B1)

$$\left.\begin{aligned} \mathbf{E}_+ = \frac{1}{2}(\mathbf{E} - Z\mathbf{u} \times \mathbf{H}), \quad \mathbf{H}_+ = \frac{1}{2}(\mathbf{H} + \frac{1}{Z}\mathbf{u} \times \mathbf{E}), \quad \text{(a),(b)} \\ \mathbf{E}_- = \frac{1}{2}(\mathbf{E} + Z\mathbf{u} \times \mathbf{H}), \quad \mathbf{H}_- = \frac{1}{2}(\mathbf{H} - \frac{1}{Z}\mathbf{u} \times \mathbf{E}), \quad \text{(c),(d)} \end{aligned}\right\} \tag{2.60}$$

and correspondingly for $\mathbf{i}_+$, $\mathbf{i}_-$, q_+, and q_- by

$$\left.\begin{aligned} \mathbf{i}_+ = \nabla \times \mathbf{H}_+ - \mu \frac{\partial \mathbf{E}_+}{\partial t}, \quad q_+ = \varepsilon \nabla \mathbf{E}_+, \\ \mathbf{i}_- = \nabla \times \mathbf{H}_- - \mu \frac{\partial \mathbf{E}_-}{\partial t}, \quad q_- = \varepsilon \nabla \mathbf{E}_-, \end{aligned}\right\} \tag{2.61}$$

where, $\mathbf{u}$ being the unit vector in the direction of $\mathbf{S}$,

$$\mathbf{u} = \frac{1}{|\mathbf{S}|}\mathbf{S}, \quad \mathbf{u}^{\mathrm{T}}\mathbf{u} = 1, \quad Z = \sqrt{\frac{\mu}{\varepsilon}}\,. \tag{2.62}$$

Using (2.9), (2.60), (2.62), and the definitions

$$\left.\begin{aligned} \mathbf{S}_+ = \mathbf{E}_+ \times \mathbf{H}_+, \quad w_+ = \frac{1}{2}(\varepsilon |\mathbf{E}_+^2| + \mu |\mathbf{H}_+^2|), \\ \mathbf{S}_- = \mathbf{E}_- \times \mathbf{H}_-, \quad w_- = \frac{1}{2}(\varepsilon |\mathbf{E}_-^2| + \mu |\mathbf{H}_-^2|), \end{aligned}\right\} \tag{2.63}$$

one finds,

$$\left.\begin{aligned} \mathbf{S}_+ &= \frac{1}{2}(\mathbf{S}+cw\mathbf{u}) = cw_+\mathbf{u}, \qquad w_+ = \frac{1}{2c}(cw+|\mathbf{S}|), \quad \text{(a),(b)} \\ \mathbf{S}_- &= \frac{1}{2}(\mathbf{S}-cw\mathbf{u}) = -cw_-\mathbf{u}, \quad w_- = \frac{1}{2c}(cw-|\mathbf{S}|). \quad \text{(c),(d)} \end{aligned}\right\} \tag{2.64}$$

In particular, $\mathbf{S}_+$ is indeed co-parallel with $\mathbf{u}$ and thus $\mathbf{S}$, while $\mathbf{S}_-$ is anti-parallel with $\mathbf{u}$ and thus $\mathbf{S}$.

From (2.64) we immediately derive for $\mathbf{S}$ and w,

$$\mathbf{S} = \mathbf{S}_+ + \mathbf{S}_-, \quad w = w_+ + w_-. \tag{2.65}$$

The simplicity of these relations is remarkable since the requirements (2.59) as well as the defining equations (2.63) are definitely non-linear. Using (2.28)(c) as well as (2.64)(b) and (d), the rest energy w_0 is found to satisfy

$$w_0 = 2\sqrt{w_+w_-} \le w_+ + w_- = w. \tag{2.66}$$

As is to be expected, w_0 vanishes if $w_+ = 0$ and/or $w_- = 0$, and, for a fixed w, it reaches its maximum for $w_+ = w_-$.

3. Field velocity and rest field: further properties

3.1 Lorentz transformation of the field velocity

As in Subsection 2.1, we consider again two reference frames, RF and RF′, with RF′ moving with constant velocity $\mathbf{v}_0$ with respect to RF. Let P be a point in RF, P′ the corresponding point in RF′, $\mathbf{v}$ the field velocity at P, and $\mathbf{v}'$ thus the field velocity in RF′. For obvious reasons we write,

$$\mathbf{v}' = (v'_x, v'_y, v'_z)^{\mathrm{T}}, \qquad \boldsymbol{\beta}' = \frac{1}{c}\mathbf{v}' = (\beta'_x, \beta'_y, \beta'_z)^{\mathrm{T}},$$

$$\alpha' = \sqrt{1-\beta'^2}, \qquad \beta'^2 = \boldsymbol{\beta}'^{\mathrm{T}}\boldsymbol{\beta}'.$$

Our goal is to determine $\mathbf{v}'$ given $\mathbf{v}$, $\boldsymbol{\gamma}$, E_0, H_0, and $\mathbf{v}_0$, assuming $\alpha_0 > 0$. A direct solution to this can be obtained by first expressing $\mathbf{v}'$ in terms of $\mathbf{E}'$ and $\mathbf{H}'$, these then in terms of $\mathbf{E}$ and $\mathbf{H}$, and the latter finally in terms of $\mathbf{v}$, $\boldsymbol{\gamma}$, E_0, and H_0. Such an approach, however, would be rather tedious, and we will therefore outline an indirect, but

simpler derivation. If $\alpha = 0$, thus if at P the field is locally planar and therefore as characterized by (2.34), it remains locally planar at P' (cf. (2.51)), implying $\alpha' = 0$, and vice versa. We therefore assume $\alpha > 0$, or equivalently $\alpha' > 0$.

For the Poynting vector at P' we then have (cf. (2.29) and (2.52)),

$$\mathbf{S}' = 2\frac{w_0'}{\alpha'^2}\mathbf{v}' = 2\frac{w_0}{\alpha'^2}\mathbf{v}' = 2c\frac{w_0}{\alpha'^2}\boldsymbol{\beta}'. \tag{3.1}$$

If however we transform $\mathbf{v}$ from P to P' according to Einstein's formulas for compounding mechanical velocities and assume (2.3) to hold, we obtain a vector $\hat{\boldsymbol{\beta}} = \hat{\mathbf{v}}/c = (\hat{\beta}_x, \hat{\beta}_y, \hat{\beta}_z)^{\mathrm{T}}$ given by

$$\left.\begin{aligned} \hat{\beta}_x &= \frac{\beta_x - \beta_0}{1 - \beta_0\beta_x} = \frac{\hat{\alpha}}{\alpha\alpha_0}(\beta_x - \beta_0), \quad \text{(a)} \\ \hat{\beta}_y &= \frac{\hat{\alpha}}{\alpha}\beta_y, \quad \hat{\beta}_z = \frac{\hat{\alpha}}{\alpha}\beta_z, \qquad \text{(b)} \end{aligned}\right\} \tag{3.2}$$

$$\hat{\alpha} = \sqrt{1 - \hat{\beta}^2}, \quad \hat{\beta}^2 = \hat{\boldsymbol{\beta}}^{\mathrm{T}}\hat{\boldsymbol{\beta}}, \quad 1 - \beta_0\beta_x = \frac{\alpha\alpha_0}{\hat{\alpha}}, \tag{3.3}$$

and we have,

$$\left.\begin{aligned} \frac{1 - \beta_0\beta_x}{\alpha^2} &= \frac{1 + \beta_0\hat{\beta}_x}{\hat{\alpha}^2} = \frac{\alpha_0}{\alpha\hat{\alpha}}, \quad \text{(a)} \\ (1 + \beta_0\hat{\beta}_x)(1 - \beta_0\beta_x) &= \alpha_0^2. \qquad \text{(b)} \end{aligned}\right\} \tag{3.4}$$

Since $\mathbf{S}$ does in fact not transform according to the Einstein velocity formulas, we have to expect that usually $\boldsymbol{\beta}' \neq \hat{\boldsymbol{\beta}}$.

In order to determine the proper expression for $\boldsymbol{\beta}'$ we first examine

$$w' = \frac{1}{2}(\varepsilon E^2 + \mu H^2) = \frac{1+\beta'^2}{\alpha'^2}w_0' = \frac{1+\beta'^2}{\alpha'^2}w_0, \tag{3.5}$$

which corresponds to (2.26) and takes into account (2.52). Using (2.5) and (2.6) we obtain

$$w' = \frac{1 + \beta_0^2}{\alpha_0^2}w - 2\frac{\beta_0}{c\alpha_0^2}S_x - \frac{\beta_0^2}{\alpha_0^2}(\varepsilon E_x^2 + \mu H_x^2),$$

and thus, eliminating $\varepsilon E_x^2 + \mu H_x^2$ by means of (2.8)(a),

$$w' + \frac{\beta_0}{c} S'_x = w - \frac{\beta_0}{c} S_x . \tag{3.6}$$

But (cf. (2.29) and (3.1)),

$$\frac{1}{c} S_x = 2 \frac{w_0}{\alpha^2} \beta_x , \quad \frac{1}{c} S'_x = 2 \frac{w'_0}{\alpha'^2} \beta'_x = 2 \frac{w_0}{\alpha'^2} \beta'_x . \tag{3.7}$$

Hence, using also (2.26) and (3.5), dividing by $2w_0$, and making use of (3.3) we can write,

$$\frac{1 - \beta_0 \beta_x}{\alpha^2} = \frac{1 + \beta_0 \beta'_x}{\alpha'^2} = \frac{\alpha_0}{\alpha \hat{\alpha}} . \tag{3.8}$$

Clearly, (3.8) is of same type as (3.4)(a).

Next we observe that the terms involving E_x and H_x in the parentheses on the right-hand sides of (2.8) are the entries 11 to 13 of the first column of $\varepsilon \mathbf{E}\mathbf{E}^{\mathrm{T}} + \mu \mathbf{H}\mathbf{H}^{\mathrm{T}}$; for this matrix we can derive after replacing $\mathbf{E}$ and $\mathbf{H}$ by (2.21) and (2.23),

$$\varepsilon \mathbf{E}\mathbf{E}^{\mathrm{T}} + \mu \mathbf{H}\mathbf{H}^{\mathrm{T}} = \frac{2w_0}{\alpha^2} \left(\boldsymbol{\gamma}\boldsymbol{\gamma}^{\mathrm{T}} + (\boldsymbol{\beta} \times \boldsymbol{\gamma})(\boldsymbol{\beta} \times \boldsymbol{\gamma})^{\mathrm{T}} \right). \tag{3.9}$$

But in view of (2.21)(b), (2.22), and (A.1),

$$(\boldsymbol{\beta} \times \boldsymbol{\gamma})(\boldsymbol{\beta} \times \boldsymbol{\gamma})^{\mathrm{T}} = \beta^2 (\mathbf{1} - \boldsymbol{\gamma}\boldsymbol{\gamma}^{\mathrm{T}}) - \boldsymbol{\beta}\boldsymbol{\beta}^{\mathrm{T}} ,$$

where $\mathbf{1}$ is the unit matrix (i.e., of order 3), and we therefore can write,

$$\varepsilon \mathbf{E}\mathbf{E}^{\mathrm{T}} + \mu \mathbf{H}\mathbf{H}^{\mathrm{T}} = \frac{2w_0}{\alpha^2} (\alpha^2 \boldsymbol{\gamma}\boldsymbol{\gamma}^{\mathrm{T}} + \beta^2 \mathbf{1} - \boldsymbol{\beta}\boldsymbol{\beta}^{\mathrm{T}}) . \tag{3.10}$$

We now replace in (2.8) the components of $\mathbf{S}$ and $\mathbf{S}'$ by means of (2.29) and (3.1), take into account (2.26) and (3.10), drop in particular the common factor $2w_0$, and make use of (3.8). The result can be put into the compact form

$$\frac{1}{\alpha'^2} \begin{pmatrix} \alpha_0 \beta'_x \\ \beta'_y \\ \beta'_z \end{pmatrix} = \frac{1}{\alpha \hat{\alpha}} (\boldsymbol{\beta} - \boldsymbol{\beta}_0) + \frac{\beta_0 \gamma_x}{\alpha_0} \boldsymbol{\gamma} \tag{3.11}$$

or, moving the term in $\boldsymbol{\beta}_0$ to the left-hand side and taking again into account (3.8),

$$\left.\begin{aligned} \frac{1}{\alpha'^2}\begin{pmatrix}\beta'_x+\beta_0\\ \alpha_0\beta'_y\\ \alpha_0\beta'_z\end{pmatrix} &= \frac{\alpha_0}{\alpha\hat{\alpha}}\boldsymbol{\beta}+\beta_0\gamma_x\boldsymbol{\gamma} && \text{(a)}\\ &= \frac{1}{\alpha^2}(1-\beta_0\beta_x)\boldsymbol{\beta}+\beta_0\gamma_x\boldsymbol{\gamma}. && \text{(b)}\end{aligned}\right\} \tag{3.12}$$

If we then eliminate β'_x between (3.8) and the first one of the three individual equations contained in (3.11) we find, writing the result directly for any choice of $\boldsymbol{\beta}_0$,

$$\left.\begin{aligned} \frac{\alpha_0^2}{\alpha'^2} &= \frac{(1-\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\beta})^2}{\alpha^2}-(\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\gamma})^2, && \text{(a)}\\ \frac{\alpha_0^2}{\alpha^2} &= \frac{(1+\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\beta}')^2}{\alpha'^2}-(\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\gamma}')^2, && \text{(b)}\end{aligned}\right\} \tag{3.13}$$

where (b) follows from (a) by replacing $\boldsymbol{\beta}_0$ by $-\boldsymbol{\beta}_0$ and interchanging primed and un-primed quantities.

Representations with the same general validity can be achieved just as easily for the combination of (3.4)(a) and (3.8) as well as for a correspondingly simpler way of writing (3.13) (a), i.e.,

$$\left.\begin{aligned} \frac{1-\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\beta}}{\alpha^2} &= \frac{1+\boldsymbol{\beta}_0^{\mathrm{T}}\hat{\boldsymbol{\beta}}}{\hat{\alpha}^2} = \frac{1+\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\beta}'}{\alpha'^2} = \frac{\alpha_0}{\alpha\hat{\alpha}}, && \text{(a)}\\ \frac{1}{\alpha'^2} &= \frac{1}{\hat{\alpha}^2}-\frac{1}{\alpha_0^2}(\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\gamma})^2. && \text{(b)}\end{aligned}\right\} \tag{3.14}$$

Furthermore, as shown in Appendix B2, we obtain from (3.11) after some manipulation,

$$\left.\begin{aligned} &\frac{\alpha_0}{\alpha'^2}\Big(\mathbf{1}-\frac{1}{1+\alpha_0}\boldsymbol{\beta}_0\boldsymbol{\beta}_0^{\mathrm{T}}\Big)\boldsymbol{\beta}' = \frac{1}{\alpha^2}(1-\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\beta})(\boldsymbol{\beta}-\boldsymbol{\beta}_0)+(\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\gamma})\boldsymbol{\gamma}, && \text{(a)}\\ &\frac{\alpha_0}{\alpha'^2}\boldsymbol{\beta}' = \Big(\mathbf{1}+\frac{1}{(1+\alpha_0)\alpha_0}\boldsymbol{\beta}_0\boldsymbol{\beta}_0^{\mathrm{T}}\Big)\left(\frac{1}{\alpha^2}(1-\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\beta})(\boldsymbol{\beta}-\boldsymbol{\beta}_0)+(\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\gamma})\boldsymbol{\gamma}\right). && \text{(b)}\end{aligned}\right\} \tag{3.15}$$

Finally, we derive from (3.8) and (3.13),

$$(1-\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\beta})(1+\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\beta}')-\alpha_0^2 = \alpha^2(\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\gamma}')^2 = \alpha'^2(\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\gamma})^2, \tag{3.16}$$

which reduces to an expression of the form (3.4)(b) (with $\boldsymbol{\beta}'$ replaced by $\hat{\boldsymbol{\beta}}$) if $\boldsymbol{\beta}_0$ is orthogonal to $\boldsymbol{\gamma}$ (and thus to $\boldsymbol{\gamma}'$, as follows from (3.16) since $\alpha > 0$ and $\alpha' > 0$).

3.2 Lorentz transformation of the electric and magnetic rest fields

In RF′, the expression corresponding to (2.36) can be written, taking into account the Lorentz invariances (2.53),

$$\boldsymbol{\gamma}' = \frac{\alpha'}{2w_0}(\varepsilon E_0 \mathbf{E}' + \mu H_0 \mathbf{H}'), \tag{3.17}$$

where α' can be determined by (3.13)(a) and, assuming again $\boldsymbol{\beta}_0$ to be given by (2.3), $\mathbf{E}'$ and $\mathbf{H}'$ thus by (2.5) and (2.6). We obtain this way from (2.36) and (3.17),

$$\begin{aligned}
\frac{1}{\alpha'}\gamma'_x &= \frac{1}{\alpha}\gamma_x, \qquad (3.18)\\
\frac{1}{\alpha'}\gamma'_y &= \frac{1}{\alpha\alpha_0}\gamma_y + \frac{\beta_0}{2c\alpha_0 w_0}(H_0 E_z - E_0 H_z),\\
\frac{1}{\alpha'}\gamma'_z &= \frac{1}{\alpha\alpha_0}\gamma_z - \frac{\beta_0}{2c\alpha_0 w_0}(H_0 E_y - E_0 H_y).
\end{aligned}$$

These expressions can further be transformed by observing that from (2.21), (2.23), and (2.24) we can deduce,

$$\alpha(H_0\mathbf{E} - E_0\mathbf{H}) = -2w_0\mathbf{v}\times\boldsymbol{\gamma}.$$

After some calculation, the desired relationship between $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$ can therefore be expressed in the form

$$\boldsymbol{\gamma}' = \mathbf{M}\boldsymbol{\gamma}, \tag{3.19}$$

where $\mathbf{M}$ is the matrix

$$\mathbf{M} = \frac{\alpha'}{\alpha\alpha_0}\begin{pmatrix} \alpha_0 & 0 & 0 \\ \beta_0\beta_y & 1-\beta_0\beta_x & 0 \\ \beta_0\beta_z & 0 & 1-\beta_0\beta_x \end{pmatrix}. \tag{3.20}$$

Since $\boldsymbol{\beta}'$ and $\boldsymbol{\gamma}'$ have the same significance in RF' as $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ do in RF, we have

$$\boldsymbol{\beta}'^{\mathrm{T}}\boldsymbol{\gamma}' = 0, \quad \boldsymbol{\gamma}'^{\mathrm{T}}\boldsymbol{\gamma}' = 1. \qquad \text{(a),(b)}\Big\}. \tag{3.21}$$

Equation (3.19) can be inverted by means of $\boldsymbol{\gamma} = \mathbf{M}'\boldsymbol{\gamma}'$, where

$$\mathbf{M}' = \frac{\alpha}{\alpha'\alpha_0}\begin{pmatrix} \alpha_0 & 0 & 0 \\ -\beta_0\beta'_y & 1+\beta_0\beta'_x & 0 \\ -\beta_0\beta'_z & 0 & 1+\beta_0\beta'_x \end{pmatrix}, \tag{3.22}$$

which is the matrix obtained from $\mathbf{M}$ by replacing everywhere β_0 by $-\beta_0$ and interchanging primed and un-primed quantities. Alternatively, we may use $\boldsymbol{\gamma} = \mathbf{M}^{-1}\boldsymbol{\gamma}'$. Although we then have

$$\mathbf{M}'\boldsymbol{\gamma}' = \mathbf{M}^{-1}\boldsymbol{\gamma}', \tag{3.23}$$

$\mathbf{M}^{-1}$ differs in general from $\mathbf{M}'$. This is easily explained by observing that $\boldsymbol{\gamma}'$ is neither independent of $\mathbf{M}'$ nor of $\mathbf{M}^{-1}$ (cf. expressions such as (2.22), (3.11) to (3.13), (3.21), and (3.22)). Obviously, a direct verification of (3.23) is tedious and will be omitted. The same is true for a direct verification of (3.21).

The desired transformation from $\mathbf{E}_0$ and $\mathbf{H}_0$ to

$$\mathbf{E}'_0 = \boldsymbol{\gamma}' E'_0, \quad \mathbf{H}'_0 = \boldsymbol{\gamma}' H'_0 \tag{3.24}$$

and vice versa can now easily be achieved. In view of (2.21), (2.53), (3.19), and (3.24) we have indeed,

$$\left.\begin{aligned} &\mathbf{E}'_0 = \mathbf{M}\mathbf{E}_0, && \mathbf{H}'_0 = \mathbf{M}\mathbf{H}_0, \\ &\mathbf{E}_0 = \mathbf{M}'\mathbf{E}'_0 = \mathbf{M}^{-1}\mathbf{E}'_0, && \mathbf{H}_0 = \mathbf{M}'\mathbf{H}'_0 = \mathbf{M}^{-1}\mathbf{H}'_0. \end{aligned}\right\} \tag{3.25}$$

3.3 Field velocity and Einstein rules for compounding velocities

We start with (3.12)(b) where we replace β_0 by $-\beta_0$ and interchange primed and un-primed quantities. This yields

$$\frac{1}{\alpha^2}\begin{pmatrix} \beta_x - \beta_0 \\ \alpha_0\beta_y \\ \alpha_0\beta_z \end{pmatrix} = \frac{1}{\alpha'^2}(1+\beta_0\beta'_x)\boldsymbol{\beta}' - \beta_0\gamma'_x\boldsymbol{\gamma}' \tag{3.26}$$

from which, making use of (3.2), (3.8), (3.18), and (3.19), one can deduce,

$$\left.\begin{aligned} &\hat{\boldsymbol{\beta}} = \boldsymbol{\beta}' + \delta'\boldsymbol{\gamma}' = \boldsymbol{\beta}' + \Delta\boldsymbol{\beta}', \quad \Delta\boldsymbol{\beta}' = \frac{\Delta\mathbf{v}'}{c} = \delta'\boldsymbol{\gamma}' = (\hat{\boldsymbol{\beta}}^{\mathrm{T}}\boldsymbol{\gamma}')\boldsymbol{\gamma}', && \text{(a),(b)} \\ &\delta' = -\frac{\alpha\hat{\alpha}}{\alpha_0}\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\gamma}' - = \frac{\alpha'\hat{\alpha}}{\alpha_0}\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\gamma} = \hat{\boldsymbol{\beta}}^{\mathrm{T}}\boldsymbol{\gamma}' = \hat{\boldsymbol{\beta}}^{\mathrm{T}}\mathbf{M}\boldsymbol{\gamma}. && \text{(c)} \end{aligned}\right\} \tag{3.27}$$

In (3.27), $\Delta\boldsymbol{\beta}'$, $\Delta\mathbf{v}'$, and δ' are defined as given, while the last two expressions for δ' can be obtained by pre-multiplying (3.27)(a) by $\boldsymbol{\gamma}'^{\mathrm{T}}$ and taking into account (3.19) and (3.21). Obviously, the expressions involving $\boldsymbol{\beta}_0$ hold in general, thus not only for $\boldsymbol{\beta}_0$ given by (2.3).

As follows from (3.27)(a), $\hat{\boldsymbol{\beta}}$ consists of two components, one of which is equal to $\boldsymbol{\beta}'$ and the other one is parallel to $\boldsymbol{\gamma}'$, thus perpendicular to $\boldsymbol{\beta}'$ (cf. (3.21)). That second component is therefore the projection of $\hat{\boldsymbol{\beta}}$ onto the unit vector $\boldsymbol{\gamma}'$ (cf. also the last expression for $\Delta\boldsymbol{\beta}'$ in (3.27)(b)). We will come back to this elegant result in Subsection 3.6.

Observe finally that from (3.27)(a) and (b) we can deduce

$$\boldsymbol{\beta}' = (\mathbf{1} - \boldsymbol{\gamma}'\boldsymbol{\gamma}'^{\mathrm{T}})\hat{\boldsymbol{\beta}},$$

but this expression cannot be inverted. In view of (A.6) we have indeed $\det(\mathbf{1} - \boldsymbol{\gamma}\boldsymbol{\gamma}^{\mathrm{T}}) = 1 - \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\gamma} = 0$.

3.4 Special case: field locally at rest

An interesting special case is a field that is locally at rest, i.e., for which

$$\boldsymbol{\beta} = \mathbf{0}, \quad \text{thus} \quad \alpha = 1 \tag{3.28}$$

at the point P under consideration. We then have $\mathbf{E}_0 = \mathbf{E}$ and $\mathbf{H}_0 = \mathbf{H}$, and therefore

$$\mathbf{E} = \boldsymbol{\gamma}E, \quad \mathbf{H} = \boldsymbol{\gamma}H, \quad \mathbf{i} = \mathbf{i}_0, \quad q = q_0.$$

Let us in particular select a Lorentz transformation for which $\boldsymbol{\beta}_0$ is also parallel to $\boldsymbol{\gamma}$. We are still free to choose the orientation of the coordinate axes, and we do this in such a way that (2.3) holds. We may then write altogether, making also use of (3.4)(a),

$$\boldsymbol{\beta}_0 = \boldsymbol{\gamma}\beta_0, \quad \boldsymbol{\gamma} = (1,0,0)^{\mathrm{T}}, \quad \hat{\alpha} = \alpha_0. \tag{3.29}$$

As a consequence of (3.28) and (3.29), the right-hand side of (3.11) vanishes, thus the left-hand side, i.e., the field at P' is also locally at rest. We therefore have at P' altogether,

$$\left.\begin{aligned} &\boldsymbol{\beta}' = \mathbf{0}, \quad \alpha' = 1, \quad \mathbf{i}' = \mathbf{i}_0', \quad q' = q_0', \quad \text{(a)} \\ &\mathbf{E}' = \mathbf{E}_0' = \boldsymbol{\gamma}'E_0', \quad \mathbf{H}' = \mathbf{H}_0' = \boldsymbol{\gamma}'H_0'. \quad \text{(b)} \end{aligned}\right\} \tag{3.30}$$

Furthermore, the matrix $\mathbf{M}$, which is defined by (3.20), simplifies in such a way that (3.19) yields,

$$\boldsymbol{\gamma}' = \boldsymbol{\gamma} = (1,0,0)^{\mathrm{T}}.$$

For $\mathbf{i}_0'$ and q_0', we conclude from (2.7) and (3.30)(a),

$$i_{0x}' = \frac{1}{\alpha_0}(i_{0x} - \beta_0 c q_0), \quad i_{0y}' = i_{0y}, \quad i_{0z}' = i_{0z},$$

$$c q_0' = \frac{1}{\alpha_0}(c q_0 - \beta_0 i_{0x}).$$

Obviously, $i_{0x}' = 0$ if $v_0 = i_{0x} / q_0$. More specifically, if $\mathbf{i}_0$ is also parallel to $\boldsymbol{\gamma}$, i.e., if it is of the form $\mathbf{i}_0 = \boldsymbol{\gamma} i_0$, we have $i_{0y} = i_{0z} = 0$ and thus altogether,

$$\mathbf{i}_0' = \mathbf{0} \quad \text{for} \quad \mathbf{v}_0 = \mathbf{v}_{ch0}, \quad \mathbf{v}_{ch0} := \frac{\mathbf{i}_0}{q_0}, \tag{3.31}$$

where $\mathbf{v}_{ch0}$, which can be interpreted as a velocity (cf. $\mathbf{v}_{ch}$ in Subsection 2.5), is defined as shown. For $\mathbf{v}_0$ to be physically valid, we must of course require $|v_0| < c$. This is satisfied by the solution (3.31) if $|\mathbf{v}_{ch0}| < c$.

We conclude from this the following: If at P we have $\boldsymbol{\beta} = \mathbf{0}$, $\mathbf{i}_0 = \boldsymbol{\gamma} i_0$, and $|\mathbf{i}_o / q_0| < c$, there exists a physically valid Lorentz transformation such that $\boldsymbol{\beta}' = \mathbf{i}_0' = \mathbf{0}$ at P'.

3.5 Lorentz transformation of rest charge and rest current

The quantities to be determined are q_0' and $\mathbf{i}_0'$ but, in contrast to Subsection 3.4, we do *not* assume $\boldsymbol{\beta} = \mathbf{0}$. We do restrict ourselves however to cases that are of particular interest for examining basal EM fields. We thus first assume $\mathbf{i}_0 = \mathbf{0}$, in which case (2.55) and therefore (2.56) hold. In view of (3.2) and (3.3), (2.56) in turn can be put into the form

$$\left.\begin{aligned} \mathbf{i}' = \frac{c q_0}{\hat{\alpha}} \hat{\boldsymbol{\beta}}, \quad q' = \frac{q_0}{\hat{\alpha}}. \quad \text{(a),(b)} \end{aligned}\right\} \tag{3.32}$$

Pre-multiplying (3.27)(a) by $\boldsymbol{\beta}'^{\mathrm{T}}$ and taking into account (3.21)(a) we obtain first from (3.27)(a) and then from (3.32)(a),

$$\left.\begin{aligned}\boldsymbol{\beta}'^{\mathrm{T}}\hat{\boldsymbol{\beta}} = \beta'^2, \quad \boldsymbol{\beta}'^{\mathrm{T}}\mathbf{i}' = cq_0\frac{\beta'^2}{\hat{\alpha}} = cq_0\frac{1-\alpha'^2}{\hat{\alpha}}. \quad \text{(a),(b)}\end{aligned}\right\} \tag{3.33}$$

Applying (2.37) to RF' and writing the result in the form

$$\mathbf{i}'_0 = \mathbf{i}' - \frac{1}{\alpha'}\left(cq' - \frac{\boldsymbol{\beta}'^{\mathrm{T}}\mathbf{i}'}{1+\alpha'}\right)\boldsymbol{\beta}', \quad cq'_0 = \frac{1}{\alpha'}(cq' - \boldsymbol{\beta}'^{\mathrm{T}}\mathbf{i}') \tag{3.34}$$

we obtain from (3.32) to (3.34),

$$\left.\begin{aligned}\mathbf{i}'_0 = \frac{cq_0}{\hat{\alpha}}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}'), \quad q'_0 = \frac{\alpha'}{\hat{\alpha}}q_0. \quad \text{(a),(b)}\end{aligned}\right\} \tag{3.35}$$

Among the various expressions that can be derived from (3.35)(a) by means of (3.26) we like to point out

$$\left.\begin{aligned}\mathbf{i}'_0 = \frac{q_0}{\hat{\alpha}}\Delta\mathbf{v}' = i'_0\boldsymbol{\gamma}', \quad i'_0 := -\frac{cq_0}{\hat{\alpha}}\delta' = -\frac{\alpha'}{\alpha_0}cq_0\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\gamma}. \quad \text{(a),(b)}\end{aligned}\right\} \tag{3.36}$$

Hence, $\mathbf{i}'_0$ is parallel to $\boldsymbol{\gamma}'$, thus to $\mathbf{E}'_0$ and $\mathbf{H}'_0$.

Next, let us replace $\mathbf{i}_0 = \mathbf{0}$ by the more general requirement

$$\mathbf{i}_0 = i_0\boldsymbol{\gamma} \tag{3.37}$$

where i_0 is a scalar (cf. also the last paragraph in Subsection 2.9). The above results (3.35)(b) and (3.36), which hold for $\mathbf{i}_0 = \mathbf{0}$, then generalize to

$$\left.\begin{aligned}\mathbf{i}'_0 = i'_0\boldsymbol{\gamma}', \quad \frac{i'_0}{\alpha'} = \frac{i_0}{\hat{\alpha}} - \frac{cq_0}{\alpha_0}\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\gamma}, \quad \text{(a),(b)}\\ \frac{q'_0}{\alpha'} = \frac{q_0}{\hat{\alpha}} - \frac{i_0}{\alpha_0 c}\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\gamma}, \qquad \text{(c)}\end{aligned}\right\} \tag{3.38}$$

where all quantities are as used so far. For proving (3.38), one can follow, broadly, the same pattern as for $\mathbf{i}_0 = \mathbf{0}$, but the calculations then needed are quite lengthy and tedious. A more elegant proof is outlined in Appendix B3. It is based on what is known as the group property of the Lorentz transformation, thus on the property that two consecutive Lorentz transformations are always equivalent to a single one.

Due to this property, the result obtained in Subsection 3.4 will indeed be seen to ensure that $\mathbf{i}'_0$ is proportional to $\boldsymbol{\gamma}'$, thus of the form (3.36)(a). In

there, the scalar i_0' still has to be determined and surely must be expected to depend on i_0.

3.6 Generalized field velocity

We continue to denote by P an arbitrary point in RF and by P' the corresponding point in RF', which itself is moving with respect to RF with constant velocity $\mathbf{v}_0$. The field velocities $\mathbf{v}$ at P and $\mathbf{v}'$ at P', say $\mathbf{v}(\mathrm{P})$ and $\mathbf{v}'(\mathrm{P}')$, are unique, and there is associated with P a rest reference frame RF_0 that moves uniformly with respect to RF with constant velocity equal to $\mathbf{v}(\mathrm{P})$. As we have seen, $\mathbf{v}'(\mathrm{P}')=\mathbf{0}$ if $\mathbf{v}_0=\mathbf{v}(\mathrm{P})$, but there arises the question as to finding all $\mathbf{v}_0$ for which $\mathbf{S}'=\mathbf{0}$, where $\mathbf{S}'$ is the Poynting vector at P'. This amounts to solving the problem defined in Subsection 2.2 after abandoning the first one of the three requirements stated there while retaining the other two. Since by definition $\boldsymbol{\beta}$ is related to $\mathbf{S}$ by (2.11), $\boldsymbol{\beta}'$ thus related correspondingly to $\mathbf{S}'$, requiring $\mathbf{S}'=\mathbf{0}$ is equivalent to requiring $\boldsymbol{\beta}'=\mathbf{0}$.

We first look for conditions that are necessary for $\mathbf{v}_0$ to be a solution of the problem with $\alpha>0$ and $\alpha_0>0$. Setting $\boldsymbol{\beta}'=\mathbf{0}$, thus $\alpha'=1$, we find from (3.14)(a), $1-\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\beta}=\alpha^2$, in particular thus

$$\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\beta}=\beta^2, \tag{3.39}$$

and hence from (3.15)(a),

$$\boldsymbol{\beta}_0=\boldsymbol{\beta}+\delta\boldsymbol{\gamma},\quad \delta=\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\gamma}. \tag{3.40}$$

Due to (2.21b) and (2.22), we thus have

$$\beta_0^2=\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\beta}_0=(\boldsymbol{\beta}^{\mathrm{T}}+\delta\boldsymbol{\gamma}^{\mathrm{T}})(\boldsymbol{\beta}+\delta\boldsymbol{\gamma})=\beta^2+\delta^2, \tag{3.41}$$

or else $\delta^2=\alpha^2-\alpha_0^2<\alpha^2$, altogether therefore

$$\boldsymbol{\beta}_0=\boldsymbol{\beta}+\delta\boldsymbol{\gamma},\quad |\delta|<\alpha.\quad \text{(a),(b)}\} \tag{3.42}$$

Vice versa, let $\boldsymbol{\beta}_0$ be given according to (3.42). Due to (2.21)(b) and (2.22), (3.41) then holds again. We thus have $\alpha_0^2=\alpha^2-\delta^2>0$, as needed, and $\alpha^2=\alpha_0^2+\delta^2\geq\alpha_0^2$, where the first inequality follows from the restriction assumed for δ, and altogether therefore

$$\alpha\geq\alpha_0>0. \tag{3.43}$$

Again due to (2.21)(b) and (2.22), pre-multiplication of the equality in (3.42) by $\boldsymbol{\beta}^{\mathrm{T}}$ and $\boldsymbol{\gamma}^{\mathrm{T}}$ yields

$$\left.\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\beta}=\beta^2, \quad \delta=\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\gamma}. \quad \text{(a),(b)}\right\} \tag{3.44}$$

Hence, if we take into account (3. 42)(a) and (3.44) the right-hand side of (3.15)(b) reduces to zero, whence indeed $\boldsymbol{\beta}'=\mathbf{0}$. In other words, (3.42) is not only necessary but also sufficient to solve the given problem with $\alpha>0$ and $\alpha_0>0$.

Let us now replace the earlier restriction $|\delta|<\alpha$ by $|\delta|\leq\alpha$. Due to (3.41), if the limit $|\delta|=\alpha$ is reached we obtain $\beta_0^2=\beta^2+\alpha^2=1$. If then $\alpha>0$, the vector $\boldsymbol{\gamma}$ is still properly defined (cf. (2.18) and (2.21)(a)). This is no longer the case if $\alpha=0$ (which had formerly been excluded by $|\delta|<\alpha$), but then necessarily also $\delta=0$ (cf. (3.41)), thus $\boldsymbol{\beta}_0=\boldsymbol{\beta}$ (cf. (3.40)) and therefore also $\beta_0^2=\beta^2=1$. The fact that the unit vector $\boldsymbol{\gamma}$ is then no longer precisely defined is irrelevant since all present results are acceptable if we interpret them as limits. In that sense, we may now replace (3.42) by

$$\boldsymbol{\beta}_0=\boldsymbol{\beta}+\delta\boldsymbol{\gamma}, \quad |\delta|\leq\alpha. \tag{3.45}$$

It is appropriate to denote by $\mathbf{v}_g$ any velocity $\mathbf{v}_0$ that reduces $\mathbf{v}'$ to zero and to call it a *generalized field velocity*. We include the limits we have just discussed and can thus write,

$$\boldsymbol{\beta}_g=\frac{1}{c}\mathbf{v}_g=\boldsymbol{\beta}+\delta\boldsymbol{\gamma}, \quad |\delta|\leq\alpha. \tag{3.46}$$

Hence, $\boldsymbol{\beta}_g$ can be decomposed into a sum of two vectors, one identical to $\boldsymbol{\beta}$ and one parallel to $\boldsymbol{\gamma}$, thus perpendicular to $\boldsymbol{\beta}$ (cf. (2.22)). It is convenient to introduce a *velocity equivalence class* VE that is associated with P in RF, and to define it as the set of all $\boldsymbol{\beta}_g$ given by (3.46), where the vectors $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are as known from the field at P. It follows from (2.22) and (3.46) that $|\boldsymbol{\beta}_g|\leq 1$, as it should be. Those $\boldsymbol{\beta}_g$ for which $|\boldsymbol{\beta}_g|<1$ and therefore $\delta^2=|\boldsymbol{\beta}_g|^2-\beta^2<1-\beta^2=\alpha^2$, thus $|\delta|<\alpha$, may be called *proper members* of VE.

3.7 Velocity equivalence class and the Einstein rules for compounding velocities

Let now $\mathbf{v}_0$ be again of general type, thus not necessarily such that $\mathbf{v}' = \mathbf{0}$, and let VE' be the velocity equivalence class associated with P' in RF'. As known from relativity theory (and as can easily be verified), (3.2) together with $\beta^2 \le 1$ and $\beta_0^2 \le 1$ guarantee $\hat{\beta}^2 \le 1$, while from $\hat{\boldsymbol{\beta}} = \boldsymbol{\beta}' + \delta'\boldsymbol{\gamma}'$ (cf. (3.27)(a)), we obtain, using (3.21), $\hat{\beta}^2 = \beta'^2 + \delta'^2$, thus for the scalar δ',

$$\delta'^2 = \hat{\beta}^2 - \beta'^2 \le 1 - \beta'^2 = \alpha'^2 .$$

Hence, $\hat{\boldsymbol{\beta}}$, which itself derives from $\boldsymbol{\beta}$ (cf. (3.2) and (3.3)), belongs indeed to VE'.

3.8 Some properties for $v_0 \to c$

We consider a field that we suppose to be known in RF'. Let $\boldsymbol{\beta}' \neq \mathbf{0}$ and $|\beta'| < 1$. In order to avoid unnecessary complications, we may also assume β'_x not to change sign in RF', and we may therefore choose the orientation of the x-axis in such a way that $\beta'_x \ge 0$. Altogether we thus assume

$$\beta'_x \ge 0, \quad 0 < \alpha' < 1, \tag{3.47}$$

but no further restriction is imposed, neither for β'_x nor for β'_y or β'_z. We observe the field in the reference frame RF with respect to which RF' moves with velocity $\mathbf{v}_0$ as given by (2.3). We are interested in the values of β_x, β_y, and β_z for $v_0 \to c$.

From (3.13)(b) we deduce,

$$\alpha^2 = \frac{\alpha_0^2 \alpha'^2}{(1 + \beta_0 \beta'_x)^2 - \alpha'^2 \beta_0^2 \gamma'^2_x} \tag{3.48}$$

For $\beta_0 \to 1$, the denominator in (3.48) tends to the limit

$$(1 + \beta'_x)^2 - \alpha'^2 \gamma'^2_x . \tag{3.49}$$

Since due to (3.47) we have $(1 + \beta'_x)^2 \ge 1$ and $\alpha'^2 \gamma'^2_x < 1$, the limit (3.49) cannot vanish, while the limit of α_0 is zero. We can therefore state:

$$\lim_{\beta_0\to 1}\alpha = 0, \quad \text{i.e.,} \quad \lim_{\beta_0\to 1}|\boldsymbol{\beta}| = 1, \quad \lim_{\beta_0\to 1}|\mathbf{v}| = c\,. \tag{3.50}$$

On the other hand, the first individual equation in (3.26) yields,

$$\beta_x = \beta_0 + \frac{\alpha^2}{\alpha'^2}\left((1+\beta_0\beta'_x)\beta'_x - \alpha'\beta_0\gamma'^2_x\right).$$

In view of (3.50) we thus obtain,

$$\lim_{\beta_0\to 1}\beta_x = 1, \quad \lim_{\beta_0\to 1}\beta_y = \lim_{\beta_0\to 1}\beta_z = 0\,. \tag{3.51}$$

We thus arrive at an important conclusion: Irrespective of the specific values of $\boldsymbol{\beta}'$ (provided the very mild assumptions $\beta'_x \geq 0$ and $\alpha' > 0$ are fulfilled), the limit of the velocity observed in RF is always equal to $\boldsymbol{\beta} = (1,0,0)^{\mathrm{T}}$. In particular, whatever the lateral components β'_y and β'_z may be, the corresponding lateral components v_y and v_z always vanish in the limit, while the longitudinal component always reaches the speed of light. This will be seen to be of importance for interpreting the spin of a photon.

4. Mechanistic properties of the electromagnetic field

4.1 Flow equations

From Maxwell's equations, thus from (2.1) and (2.2), two further useful equations can be derived, which, in our notation, can be written as follows:

$$\left.\begin{aligned} &\frac{\partial \mathbf{j}}{\partial t} + (\nabla^{\mathrm{T}}\mathbf{T}_c)^{\mathrm{T}} + \mathbf{f}_c = \mathbf{0}, \quad \text{(a)} \\ &\frac{\partial w}{\partial t} + \nabla^{\mathrm{T}}\mathbf{S} + \mathbf{i}^{\mathbf{T}}\mathbf{E} = 0. \quad \text{(b)} \end{aligned}\right\} \tag{4.1}$$

To these we add the original equations defining q and $\mathbf{i}$, thus

$$\left.q = \varepsilon\nabla^{\mathrm{T}}\mathbf{E}, \quad \mathbf{i} = \nabla\times\mathbf{H} - \varepsilon\frac{\partial \mathbf{E}}{\partial t}. \quad \text{(a),(b)}\right\} \tag{4.2}$$

In (4.1)(a),

$$\mathbf{j} = \frac{1}{c^2}\mathbf{S} = \frac{1}{c^2}\mathbf{E}\times\mathbf{H} \tag{4.3}$$

is the classically known *momentum density* of the field,

$$\mathbf{f}_c = q\mathbf{E} + \mu\mathbf{i}\times\mathbf{H} \tag{4.4}$$

the *classical Lorentz force density*, and $\mathbf{T}_c$, i.e.,

$$\mathbf{T}_c = w\mathbf{1} - \varepsilon\mathbf{E}\mathbf{E}^{\mathrm{T}} - \mu\mathbf{H}\mathbf{H}^{\mathrm{T}}, \tag{4.5}$$

a classical stress tensor equal to the negative of what is known as *Maxwell's stress tensor*, while w, $\mathbf{S}$, and c are as defined throughout this paper. While (4.1)(b) is commonly encountered, (4.1)(a) and (4.5) are less well known. Note that the systematic use of the transposition operator has made it possible to present these equations in a very compact form.

Together, (4.1) and (4.2), to which should be added the defining equations (2.9) and (4.3) to (4.5), are *essentially* equivalent to the original equations (2.1) and (2.2), i.e., equivalent at least in all those points P that have no neighbourhood in which $\mathbf{E}^{\mathrm{T}}\mathbf{H}$ vanishes everywhere (cf. Appendix B4).

Although it is standard to interpret $\mathbf{j}$ in (4.1)(a) as a momentum density and $\mathbf{T}_c$ as a measure for surface forces, a fully satisfactory justification for this is not feasible and can therefore not be found in the literature. The equations (4.1) should indeed be compared to the corresponding conservation equations of fluid dynamics, i.e., $\mathbf{v}$ being the velocity of the fluid,

$$\left.\begin{aligned} &\frac{\partial \mathbf{j}}{\partial t} + (\nabla^{\mathrm{T}}(\mathbf{v}\mathbf{j}^{\mathrm{T}}))^{\mathrm{T}} + (\nabla^{\mathrm{T}}\mathbf{T})^{\mathrm{T}} + \mathbf{f}_g = \mathbf{0}, \quad \text{(a)} \\ &\frac{\partial w}{\partial t} + \nabla^{\mathrm{T}}(w\mathbf{v}) + \nabla^{\mathrm{T}}(\mathbf{T}\mathbf{v}) + \mathbf{v}^{\mathrm{T}}\mathbf{f}_g = 0, \quad \text{(b)} \end{aligned}\right\} \tag{4.6}$$

which concern the rate of change of the momentum density $\mathbf{j}$ and the energy density w of the fluid. In (4.6)(a), the term $(\nabla^{\mathrm{T}}\mathbf{v}\mathbf{j}^{\mathrm{T}})^{\mathrm{T}}$ is due to the fact that the momentum density $\mathbf{j}$, which is proportional to the velocity $\mathbf{v}$, is itself moving along with the same velocity $\mathbf{v}$. Furthermore, the volume and the surface force densities are represented, respectively, by the vector $\mathbf{f}_g$ (usually due to gravitation) and the matrix (tensor) $\mathbf{T}$, which is determined by the pressure and the viscosity forces. The corresponding terms in (4.6)(b) describe the work done by these forces, while $\nabla^{\mathrm{T}}(w\mathbf{v})$ is due to the convection of the energy density.

A proper analogy between (4.1) and (4.6), clearly, is not possible. On the one hand, the term $(\nabla^{\mathrm{T}}\mathbf{v}\mathbf{j}^{\mathrm{T}})^{\mathrm{T}}$ is missing in (4.1)(a). In order to justify

the interpretation of $\mathbf{j}$ one cannot therefore, for instance, simply integrate (4.1)(a) over an arbitrary volume V of finite extent and delimited by a surface F and then apply Gauss' theorem. It is therefore customary to argue by considering the limit when F goes to infinity and to assume that the field on F vanishes sufficiently fast so that the total flux of $\mathbf{j}$ across F goes to zero.

In (4.1)(b), on the other hand, the term in $\mathbf{S}$ assumes a role similar to that of the convection term (second term) in (4.6)(b), but while the work done by the volume forces is taken into account by the third term, there is no corresponding term involving the surface forces, or else, (4.1)(b) suggests that the electromagnetic surface forces cannot do any work. As will be seen, these dilemmas are fully overcome if we make use of the concepts developed in the previous sections.

Let us thus make use of the relevant results in Section 2 and, to a small extent, Section 3. We obtain from (4.3) together with (2.26), (2.29), (2.48), and (2.49),

$$\mathbf{j} = \frac{1}{c^2}\mathbf{S} = \frac{2w_0}{c^2\alpha^2}\mathbf{v} = \frac{w_0}{c^2\alpha_c}\mathbf{v}_c\,, \tag{4.7}$$

and from (3.10), (4.5), and (4.7),

$$\mathbf{T}_c = \mathbf{T}_0 + \mathbf{v}\mathbf{j}^{\mathrm{T}}\,, \tag{4.8}$$

where
$$\mathbf{T}_0 = \mathbf{T}_0^{\mathrm{T}} := w_0\mathbf{U}, \quad \mathbf{U} = \mathbf{U}^{\mathrm{T}} := (\mathbf{1} - 2\boldsymbol{\gamma}\boldsymbol{\gamma}^{\mathrm{T}})\,. \tag{4.9}$$

Furthermore, defining w_i by

$$w_i = \frac{w_0}{\alpha^2} \tag{4.10}$$

we derive from (2.26),

$$\left.\begin{aligned} 2w_i = w_0 + w, \quad w = (1+\beta^2)w_i = \frac{1+\beta^2}{\alpha^2}w_0\,, \quad \text{(a),(b)} \end{aligned}\right\} \tag{4.11}$$

and thus from again (2.29),

$$\mathbf{S} = 2w_i\mathbf{v} = \frac{2w_0}{\alpha^2}\mathbf{v} = w\mathbf{v} + w_0\mathbf{v} = w\mathbf{v} + \mathbf{T}_0\mathbf{v}\,. \tag{4.12}$$

For obtaining this result we have made use of

$$\mathbf{T}_0\mathbf{v} = w_0\mathbf{v}\,, \tag{4.13}$$

which follows from (2.22) and (4.9). Finally, taking into account (4.8) and (4.12), (4.1) can be rewritten in the following way:

$$\left.\begin{aligned} &\frac{\partial \mathbf{j}}{\partial t}+\left(\nabla^{\mathrm{T}}(\mathbf{v}\mathbf{j}^{\mathrm{T}})\right)^{\mathrm{T}}+(\nabla^{\mathrm{T}}\mathbf{T}_0)^{\mathrm{T}}+\mathbf{f}_c=\mathbf{0}, \quad \text{(a)} \\ &\frac{\partial w}{\partial t}+\nabla^{\mathrm{T}}(w\mathbf{v})+\nabla^{\mathrm{T}}(\mathbf{T}_0\mathbf{v})+\mathbf{i}^{\mathrm{T}}\mathbf{E}=0. \quad \text{(b)} \end{aligned}\right\} \tag{4.14}$$

We call these the *flow equations* of the EM field.

Clearly, (4.14)(a) and (b), which are strictly equivalent to the original (4.1)(a) and (b), have the same general structure as (4.6)(a) and (b). Thus, the terms describing convection of the momentum density $\mathbf{j}$ and the energy density w are present precisely in the way needed. The same is true for the stress tensor $\mathbf{T}_0$ that describes the surface forces and the work done by these forces. As can be seen by using (4.4) and then (2.15), (A.3), (2.18), and (2.41), the term $\mathbf{i}^{\mathrm{T}}\mathbf{E}$ in (4.14)(b) is related by

$$\mathbf{i}^{\mathrm{T}}\mathbf{E}=\mathbf{v}^{\mathrm{T}}\mathbf{f}_c+\alpha\mathbf{i}_0^{\mathrm{T}}\mathbf{E}_0 \tag{4.15}$$

to the work done by the volume force $\mathbf{f}_c$. An additional contribution is due to $\mathbf{i}_0$ and $\mathbf{E}_0$, thus to the rest field, but this becomes irrelevant in the case of a basal EM field.

The *stress tensor* $\mathbf{T}_0$, which appears in (4.14)(a) and (b), clearly differs from Maxwell's stress tensor $-\mathbf{T}_c$. As follows by comparing (4.9) with (4.5), $\mathbf{T}_0$ is equal to the expression one obtains by replacing everywhere in $\mathbf{T}_c$ the actual field by the rest field; it therefore comprises only one dyadic product instead of two. The matrix $\mathbf{U}$ in (4.9) is orthogonal and is equal to what is known in numerical mathematics as a *Householder matrix* [20]. Clearly, $\mathbf{T}_0$ may be claimed to be simpler than $\mathbf{T}_c$, but the latter reduces to the former for $\mathbf{v}=0$.

An equation corresponding to (4.14)(b) can be derived also for w_i. As shown in Appendix B5, we can indeed obtain from (4.14)(a) and (b) for w_i the relation

$$\frac{\partial w_i}{\partial t}+\nabla^{\mathrm{T}}(w_i\mathbf{v})+(\mathbf{T}_i\nabla)^{\mathrm{T}}\mathbf{v}+\frac{1}{\alpha}\mathbf{i}_0^{\mathrm{T}}\mathbf{E}_0=0 \tag{4.16}$$

where, using (4.9) and (4.10),

$$\mathbf{T}_i=\frac{1}{\alpha^2}\mathbf{T}_0=w_i\mathbf{U}=w_i(\mathbf{1}-2\boldsymbol{\gamma}\boldsymbol{\gamma}^{\mathrm{T}}). \tag{4.17}$$

Although the first two terms in (4.16) have the same general form as the corresponding ones in (4.14)(b), there is an essential difference between the third terms; this is not surprising since w_i is not the total energy density of the field. From (4.10), (4.13), (4.15), and (4.16), we can now easily derive for w_0 the relation

$$\frac{\partial w_0}{\partial t} + 2(\mathbf{T}_i\nabla)^{\mathrm{T}}\mathbf{v} + \frac{1+\beta^2}{\alpha^2}\mathbf{i}_0^{\mathrm{T}}\mathbf{E}_0 - \mathbf{v}^{\mathrm{T}}\mathbf{f}_c = 0\,, \tag{4.18}$$

while for q we have, as classically known,

$$\frac{\partial q}{\partial t} + \nabla^{\mathrm{T}}\mathbf{i} = 0\,. \tag{4.19}$$

4.2 Flow equations of a basal electromagnetic field

The above equations simplify substantially if the EM field is basal ($\mathbf{i}_0 = \mathbf{0}$ or, equivalently, $\mathbf{i} = q\mathbf{v}$, cf. Subsection 2.9). Due to (2.18)(a) and (2.55), (4.4) becomes

$$\left.\begin{aligned} &\mathbf{f}_c = q(\mathbf{E} + \mu\mathbf{v}\times\mathbf{H}) = \mathbf{f}_0\,, \quad \text{(a)} \\ &\mathbf{f}_0 := q_0\mathbf{E}_0\,, \qquad\qquad\quad \text{(b)} \end{aligned}\right\} \tag{4.20}$$

which is the same result as if we had simply evaluated (4.4) for $\mathbf{v} = \mathbf{0}$, i.e., if in (4.4) we had simply replaced q, $\mathbf{E}$, $\mathbf{i}$, and $\mathbf{H}$ by q_0, $\mathbf{E}_0$, $\mathbf{i}_0 = \mathbf{0}$, and $\mathbf{H}_0$, respectively, (although the value of $\mathbf{H}_0$ is in fact irrelevant). In turn, we now derive from (2.19), (4.15), and (4.20),

$$\mathbf{v}^{\mathrm{T}}\mathbf{f}_c = \mathbf{v}^{\mathrm{T}}\mathbf{f}_0 = 0, \quad \mathbf{i}^{\mathrm{T}}\mathbf{E} = 0\,, \tag{4.21}$$

Hence, the contribution of the volume forces to the work done vanishes for a basal field.

The equations (4.14), (4.16), (4.18), and (4.19) now read

$$\frac{\partial \mathbf{j}}{\partial t} + \left(\nabla^{\mathrm{T}}(\mathbf{v}\mathbf{j}^{\mathrm{T}})\right)^{\mathrm{T}} + (\nabla^{\mathrm{T}}\mathbf{T}_0)^{\mathrm{T}} + \mathbf{f}_0 = \mathbf{0}\,, \tag{4.22}$$

$$\frac{\partial w}{\partial t} + \nabla^{\mathrm{T}}(w\mathbf{v}) + \nabla^{\mathrm{T}}(\mathbf{T}_0\mathbf{v}) = 0\,, \tag{4.23}$$

$$\frac{\partial w_i}{\partial t} + \nabla^{\mathrm{T}}(w_i\mathbf{v}) + (\mathbf{T}_i\nabla)^{\mathrm{T}}\mathbf{v} = 0\,, \tag{4.24}$$

$$\frac{\partial w_0}{\partial t} + 2(\mathbf{T}_i\nabla)^{\mathrm{T}}\mathbf{v} = 0\,, \tag{4.25}$$

$$\frac{\partial q}{\partial t} + \nabla^{\mathrm{T}}(q\mathbf{v}) = 0\,. \tag{4.26}$$

Furthermore, q can now be eliminated between (4.2)(a) and (b), yielding

$$\varepsilon\frac{\partial \mathbf{E}}{\partial t} + \varepsilon\mathbf{v}\nabla^{\mathrm{T}}\mathbf{E} = \nabla\times\mathbf{H}\,. \tag{4.27}$$

To these equations have to be added (2.55)(a), (4.2)(a), (4.10), and (4.20)(b), and furthermore equations such as (2.22) to (2.24), which do indeed not explicitly contain $\mathbf{i}_0$ and therefore keep their original appearance.

The results thus obtained are remarkable, as will become more evident in the course of our further analysis. Note that the contributions by the surface and the volume forces in (4.22) depend only on the rest field. On the other hand, as a result of (4.21), and as is in fact visible in (4.23), volume forces do not contribute to the energy balance, contrary to the surface forces. This expresses that in a basal field energy can be transmitted only by convection (second term in (4.23)) and by work done by the surface forces.

As (4.23) shows, $\partial w/\partial t$ is affected by the convection term $\nabla^{\mathrm{T}}(w\mathbf{v})$ and by the term $\nabla^{\mathrm{T}}(\mathbf{T}_0\mathbf{v})$, which takes into account the work done by the surfaces forces and, in view of (4.13), is in fact equal to $\nabla^{\mathrm{T}}(w_0\mathbf{v})$. Due for instance to (2.46) and (4.12) we can write,

$$w\mathbf{v} + \mathbf{T}_0\mathbf{v} = (w + w_0)\mathbf{v} = w\mathbf{v}_c\,. \tag{4.28}$$

Hence, the changes w undergoes due on the one hand to convection and, on the other, to the work done by the forces can be combined into a single energy transport phenomenon that takes place with an *equivalent velocity* $\mathbf{v}_c$ equal to the classical energy velocity (cf. Subsection 2.6). In particular, (4.23) can this way be replaced by the more compact expression

$$\frac{\partial w}{\partial t} + \nabla^{\mathrm{T}}(w\mathbf{v}_c) = 0\,. \tag{4.29}$$

The interest in using $\mathbf{v}_c$ is further enhanced by recalling the expression $w = w_0 / \alpha_c$ (cf. (2.49)) and by writing the momentum density $\mathbf{j}$ in the form

$$\mathbf{j} = \frac{1}{c^2} w \mathbf{v}_c , \tag{4.30}$$

as follows directly from (2.46) and (4.3).

Nevertheless, according to the second term in (4.22), the convection flow of $\mathbf{j}$, just like that of ρ according to (4.26), does take place with velocity $\mathbf{v}$, *not* with $\mathbf{v}_c$. A fully meaningful analogy with fluid dynamics is therefore impossible if we would attempt to build it on the basis of $\mathbf{v}_c$ instead of $\mathbf{v}$. In fluid dynamics, indeed, both the velocity by which the fluid moves and that by which the resulting momentum density is transported, are identical. This is the case for an EM field if in (4.14) and (4.22) we express $\mathbf{j}$ in terms of $\mathbf{v}$. Hence, there is no way of properly interpreting, for instance, (4.22) by simply appealing to w and $\mathbf{v}_c$.

We could clearly have made use of (4.28) also for fields that are not basal, but any attempt to find a proper interpretation then faces a serious obstacle: The contribution due to the work done by the volume forces does no longer vanish, but it cannot be combined with that of the surface forces either and a proper definition of an equivalent velocity is therefore impossible.

4.3 Electromagnetic fluid and relativistic interpretation

With the momentum density written in the form

$$\left.\begin{aligned} \mathbf{j} = m_i \mathbf{v}, \quad m_i = \frac{m_0}{\alpha^2} = \frac{2}{c^2} w_i , \quad \text{(a),(b)} \\ w_i = \frac{1}{2} m_i c^2 = \frac{w_0}{\alpha^2} , \qquad \text{(c)} \end{aligned}\right\} \tag{4.31}$$

where m_0 and w_0 are related by

$$m_0 = m_i\big|_{\mathbf{v}=\mathbf{0}} = \frac{2}{c^2} w_0 , \quad w_0 = w\big|_{\mathbf{v}=\mathbf{0}} = w_i\big|_{\mathbf{v}=\mathbf{0}} = \frac{1}{2} m_0 c^2 , \tag{4.32}$$

the equations (4.22) and (4.23) assume precisely the structure of the basic equations of fluid dynamics, the Navier-Stokes equations (see also Subsection 4.4). We may therefore rightfully speak of an *electromagnetic fluid* (*EM fluid*). At the *basic*, or *primary*, *level* of observation, the essence of the behaviour of that EM fluid is indeed described by (4.22), (4.23), and (4.31), m_i being some mass density, say the *inertial mass density*, and $\mathbf{v}$ the local velocity.

At first glance, such an attempt to interpret the flow equations seems to be incompatible with basic results from *classical relativistic dynamics* since m_0 visibly appears in the equations as a *rest mass density* and w_i as an *inertial energy density*. On the other hand, an *alternative* form of *relativistic dynamics*, which has become known more recently [13-15] and is summarized and updated in Appendix C1 and C2, offers an answer to this dilemma; its results have been obtained by several distinct approaches using analogies and logical deductions. Understandably, such an alternative theory may appear to imply an insurmountable conflict with classical special relativity, whose predictions are known to have been confirmed with high precision by many carefully performed experiments (e.g. [21-23]). As will be seen, however, a more thorough analysis reveals that in reality no such conflict exists.

Before addressing that issue more deeply, let us observe that (4.24) exhibits an obvious similarity with that specific fluid-dynamic equation that concerns the evolution of the internal energy density. Hence, w_i may equally well be interpreted as an *internal energy density*. Just like in the kinetic theory of gases, we may then consider two types of kinetic energy, an *internal kinetic energy density* w_{ki} and an *external kinetic energy density* w_{ke}, with (cf. also (4.10), (4.11)(b), and (4.31)(b) and (c)),

$$w_{ki} := w_i - w_0\,, \quad w_{ke} := w - w_i = \frac{1}{2} m_i v^2\,. \tag{4.33}$$

In a fluid, the former corresponds to the hidden kinetic energy it possesses due to the thermal agitation of its molecules (thermal energy), while the latter is due to its more immediately observable macroscopic movement. One important difference exists, however: While in a conventional fluid the two forms of kinetic energy can independently admit any reachable value, they are strictly equal in an EM fluid. As (4.11)(a) shows, we have indeed,

$$w_{ke} = w - w_i = w_i - w_0 = w_{ki}\,,$$

and therefore for the *total kinetic energy* w_k,

$$\left.\begin{aligned} w_k = w_{ki} + w_{ke} = 2w_{ki} = 2w_{ke} = m_i v^2 \quad &\text{(a)} \\ = w - w_0 = 2\frac{\beta^2}{\alpha^2} w_0\,. \quad &\text{(b)} \end{aligned}\right\} \tag{4.34}$$

In order to prepare the ground for resolving the mentioned conflict between classical and alternative relativistic dynamics, we consider next the *secondary level* at which the phenomena can be observed. At this (in a sense: dependent) level the two entirely different mechanisms producing changes in w are merged into a single effect that amounts to a resultant energy flow taking place with velocity equal to $\mathbf{v}_c$, as has been explained above (c.f. (4.28) to(4.30)). With $\mathbf{j}$ being strictly the same quantity as in (4.31)(a), we can now write

$$\left.\mathbf{j} = m_i\mathbf{v} = m_c\mathbf{v}_c, \quad m_c = \frac{m_{c0}}{\alpha_c}, \quad \text{(a),(b)}\right\} \tag{4.35}$$

$$\left.\begin{aligned} w_{c0} &= w\big|_{\mathbf{v}_c=\mathbf{0}} = m_{c0}c^2 = \frac{1}{2}m_0c^2 = w_0, && \text{(a)}\\ m_{c0} &= m_c\big|_{\mathbf{v}_c=\mathbf{0}} = \frac{w_0}{c^2} = \frac{1}{2}m_0, && \text{(b)}\end{aligned}\right\} \tag{4.36}$$

$$\left.\begin{aligned} w = m_cc^2 &= \frac{w_0}{\alpha_c} = \frac{1+\beta^2}{\alpha^2}w_0 = (1+\beta^2)w_i, && \text{(a)}\\ 2m_c &= (1+\beta^2)m_i, && \text{(b)}\end{aligned}\right\} \tag{4.37}$$

where $\mathbf{v}_c$, w, and α_c are related to $\mathbf{v}$, w_i, and α by means of (2.47) to (2.49) and (4.11), and where both w_0 and m_0 have the same meaning as before.

As will be confirmed by the results of Sections 5 and 6, it is reasonable to assume at this point that EM particles (electrons/positrons, photons) are condensed basal EM fields. They then have an inner structure that is fully governed by Maxwell's equations, yet behaves like a fluid, as has been explained. The fine details of the dynamics in that EM fluid follow laws that can be explained by means of the alternative form of relativistic dynamics (cf. Appendix C3).

As far as the overall flow of the energy density is concerned, however, thus at the secondary level, laws of classical relativistic dynamics come into play. Indeed, the relevant results in (4.35) to (4.37) such as $\mathbf{j} = m_c\mathbf{v}_c$ and $w = w_0/\alpha_c = m_cc^2$ are perfectly compatible with the corresponding ones in classical relativity. Since m_{c0} is only half as large as m_0 while $\mathbf{j}$

is unaffected by the choice of the level at which the phenomena are observed, there is now no factor $1/2$ multiplying m_{c0} in $w_{c0} = w_0 = m_{c0}c^2$, and the divisor α^2 in $m_i = m_0/\alpha^2$ is replaced by α_c in $m_c = m_{c0}/\alpha_c$. On the other hand, the ratios w/w_i and $2m_c/m_i$ increase monotonically from 1 at small velocities to 2 at the speed of light, while (cf. (2.48)) the ratio $|\mathbf{v}_c|/|\mathbf{v}|$ decreases simultaneously from 2 to 1. In line with this, $|\mathbf{j}|$ increases from 0 to

$$|\mathbf{j}| = m_c c = \frac{w}{c} \quad \text{for} \quad |\beta| = 1, \tag{4.38}$$

thus at the speed of light.

If an EM particle is moving as a whole, thus as some entity of its own, its internal structure is irrelevant. All that counts, obviously, is the resultant overall movement of its energy. From what we have seen about movement of energy density (second level of observation), it is thus not surprising that the dynamics of an EM particle (third level of observation) does indeed follow the laws of classical relativity, as will be confirmed in Section 7. This explains why experiments made with EM particles yield results compatible with that long established theory.

In fact, it can be shown that there are only two reasonably simple ways of establishing a dynamics theory compatible with the requirements of the Lorentz transformation and therefore with relativistic kinematics, which indeed remains valid for the alternative theory as for the classical one. Any other approach that at first sight might look feasible becomes quickly very intricate and unappealing and thus unlikely to be of any relevance. In any case, it is not surprising to see the only two acceptable solutions play a definite role in describing the actual phenomena: The alternative one is relevant for the behaviour of the hidden inner structure, thus for the dynamics at the very basic level of what can, at least in principle, be observed, while the classical one describes the behaviour at the more easily observable third level, which in a sense corresponds to the macroscopic behaviour of an EM particle.

4.4 Beyond Maxwell's equations

All results we have obtained so far have followed strictly from Maxwell's equations. This is true for EM fields that are basal as well as for more general, thus non-basal ones. As we have seen, equations such as

(4.14), (4.22), and (4.23) are of the same type as corresponding equations of fluid dynamics. Since (4.14)(a) and (4.22) imply that the field has inertia, thus mass, it is clear that Maxwell's equations must be generalized in order to take into account the presence of forces that are not of EM nature but act directly upon a mass density, in particular thus the forces due to gravitation. This holds regardless of whether the gravitational forces are due to the EM field itself or are imposed from the outside.

Equations such as (4.14)(a) and (4.22) point into the direction in which a required generalization might be attempted. The simplest would be to add to these equations
further force-density terms that take into account the presence of the gravitational field $\mathbf{G}$. Equations such as (4.14)(b) and (4.23) must then be modified correspondingly. It is true that gravitational forces are usually far smaller than EM forces and therefore totally negligible in an EM context, but it is likely that there are situations in which this is no longer permitted.

Of course, equations such as (4.14) or (4.22) and (4.23) are still insufficient for a complete description of an EM field. They must therefore be supplemented by whatever further equations are needed in order to arrive at a complete set, which in turn must be equivalent to the original set given by (2.1) and (2.2) (or proper generalizations thereof). However, further investigations into this challenging topic are beyond the scope of the present text.

5. Localized basal electromagnetic fields: rotating field

5.1 Nonlinear partial differential equations describing the field

We call an electromagnetic (EM) field *localized* if at any time instant its charge and its energy are essentially concentrated in a volume of small size. In Section 2, the concept of field velocity has been introduced and clearly defined. This allows us to consider hereafter a basal EM field that is localized and rotating around an axis. We assume this field in particular to have circular symmetry about its axis and to have appropriate symmetry with respect to an equatorial plane. We use standard spherical coordinates r, θ, and. A vector, say $\mathbf{a}$, can then be represented in terms of its spherical components by writing (cf. Appendix A2),

$$\mathbf{a} = (a_r, a_\theta, a_\varphi)^{\mathrm{T}}.$$

We assume the rotation to be steady, i.e. independent of time.

To be precise, we make altogether the following *assumptions*:
1. *All* quantities are independent of φ but may depend on r, and θ.
2. We have $v_r = v_\theta = i_r = i_\theta = 0$, i.e., using the simplifying notation $v = v_\varphi$ and $i = i_\varphi$,

$$\mathbf{v} = (0,0,v)^{\mathrm{T}}, \quad \mathbf{i} = (0,0,i)^{\mathrm{T}}. \tag{5.1}$$

3. Both v and i are symmetric with respect to the equatorial plane determined by $\theta = \pi/2$.
4. The field is basal in the given reference frame.
5. The rotation is steady, i.e., all quantities are independent of t.

Some consequences can be drawn from these assumptions:
1. Due to (2.15), (2.19), (2.22), and (5.1), we have

$$E_\varphi = E_{0\varphi} = H_\varphi = H_{0\varphi} = \gamma_\varphi = 0. \tag{5.2}$$

2. In view of the 4th assumption we may write,

$$\mathbf{i}_0 = \mathbf{0}, \quad \mathbf{i} = q\mathbf{v}, \quad i = qv. \tag{5.3}$$

3. The angular frequency ω defined by

$$v = \omega r \sin\theta$$

is not required to be constant; it may thus be a function of r and θ, but not of φ.

For the further analysis, we could now proceed from (4.3) and (4.14), but care must then be exercised because, in the form written, (4.14) holds only for Cartesian coordinates. We therefore use the original equations (2.1) and (2.2), for which the representation in terms of spherical coordinates is commonly known (Appendix A2). Taking into account the above assumptions and their subsequent consequences, 4 of the original 8 individual equations in (2.1) and (2.2) are found to be identically satisfied. Two of them, i.e.

$$r\frac{\partial H_\theta}{\partial r} + H_\theta - \frac{\partial H_r}{\partial \theta} = qvr, \tag{5.4}$$

$$r\frac{\partial E_r}{\partial r} + 2E_r + \frac{\partial E_\theta}{\partial \theta} + E_\theta \cot\theta = \frac{1}{\varepsilon} qr, \tag{5.5}$$

contain q, which can therefore be eliminated. We thus deduce altogether the following individual equations:

$$r\frac{\partial E_\theta}{\partial r}+E_\theta-\frac{\partial E_r}{\partial\theta}=0\,, \tag{5.6}$$

$$r\frac{\partial H_r}{\partial r}+2H_r+\frac{\partial H_\theta}{\partial\theta}+H_\theta\cot\theta=0\,, \tag{5.7}$$

$$\varepsilon v(r\frac{\partial E_r}{\partial r}+2E_r+\frac{\partial E_\theta}{\partial\theta}+E_\theta\cot\theta)=r\frac{\partial H_\theta}{\partial r}+H_\theta-\frac{\partial H_r}{\partial\theta}\,. \tag{5.8}$$

For $\mathbf{E}$ and $\mathbf{H}$ in terms of $\mathbf{E}_0$ and $\mathbf{H}_0$ we obtain from (2.23) and the relevant individual equalities in (5.1) and (5.2),

$$\left.\begin{aligned} E_r&=\frac{1}{\alpha}(E_{0r}+\mu v H_{0\theta}),\quad E_\theta=\frac{1}{\alpha}(E_{0\theta}-\mu v H_{0r}),\quad \text{(a)}\\ H_r&=\frac{1}{\alpha}(H_{0r}-\varepsilon v E_{0\theta}),\quad H_\theta=\frac{1}{\alpha}(H_{0\theta}+\varepsilon v E_{0r}).\quad \text{(b)}\end{aligned}\right\} \tag{5.9}$$

In view of (2.20), the components of the rest field are not independent but must be such that

$$E_{0r}H_{0\theta}=E_{0\theta}H_{0r}\,. \tag{5.10}$$

The other two equations implied by (2.20) are identically satisfied. Inverting (5.9) yields

$$\left.\begin{aligned} E_{0r}&=\frac{1}{\alpha}(E_r-\mu v H_\theta),\quad E_{0\theta}=\frac{1}{\alpha}(E_\theta+\mu v H_r),\quad \text{(a)}\\ H_{0r}&=\frac{1}{\alpha}(H_r+\varepsilon v E_\theta),\quad H_{0\theta}=\frac{1}{\alpha}(H_\theta-\varepsilon v E_r),\quad \text{(b)}\end{aligned}\right\} \tag{5.11}$$

while substituting (5.11) in (5.10) and taking into account (5.2) results in

$$S_\varphi=E_rH_\theta-E_\theta H_r\,,\quad S_r=S_\theta=0,$$

thus, using (2.11)(a), in

$$\mathbf{S}=(0,0,S)^{\mathrm{T}}\,,\quad S:=S_\varphi=\frac{2vw}{1+\beta^2}\,, \tag{5.12}$$

where

$$2w=\varepsilon(E_r^2+E_\theta^2)+\mu(H_r^2+H_\theta^2)\,. \tag{5.13}$$

Clearly, (5.12) and (5.13) are as required by (2.11) and (2.9), respectively. We assume of course that E_r, E_θ, H_r, and H_θ vanish sufficiently fast at infinity, and the field may then indeed be considered to be localized in the sense defined above.

In accordance with the discussion given at the end of Subsection 2.3, we can either choose any v and any E_{0r}, $E_{0\theta}$, H_{0r}, and $H_{0\theta}$ that satisfy (5.10), or any E_r, E_θ, H_r, and H_θ but with v then satisfying (5.12) and (5.13). If we adopt the first one of these two alternatives and substitute (5.9) in (5.6) to (5.8) we obtain 3 new PDEs, to which (5.10) must be added. We then have altogether 4 equations in the 5 unknowns v, E_{0r}, $E_{0\theta}$, H_{0r}, and $H_{0\theta}$, thus one equation less than the number of unknowns. Hereafter, we adopt a point of view closer to the second alternative.

5.2 Use of normalized quantities

As will be seen, it is helpful to make use of various normalized quantities. They are obtained by multiplying the original quantity by a constant factor such that the resulting quantity is dimensionless. Except for $\beta = v/c$, they will be denoted by placing a circumflex (hat) over the original symbol.

Accordingly, we start by defining

$$\hat{E}_r := \sqrt{\frac{\varepsilon}{W_n}} E_r, \quad \hat{E}_\theta := \sqrt{\frac{\varepsilon}{W_n}} E_\theta ,$$

$$\hat{H}_r := \sqrt{\frac{\mu}{W_n}} H_r, \quad \hat{H}_\theta := \sqrt{\frac{\mu}{W_n}} H_\theta ,$$

$$\hat{q} := \frac{r_n}{\sqrt{\varepsilon W_n}} q, \quad \hat{S}_\varphi := \frac{1}{cW_n} S_\varphi, \quad \hat{r} := \frac{r}{r_n},$$

where W_n and r_n are arbitrary normalizing constants having the dimension of an energy density and a length, respectively. We thus deduce from (5.6) to (5.8) as well as (5.12) and (5.13) the equations

$$\hat{r}\frac{\partial \hat{E}_\theta}{\partial \hat{r}} + \hat{E}_\theta - \frac{\partial \hat{E}_r}{\partial \theta} = 0, \tag{5.14}$$

$$\hat{r}\frac{\partial \hat{H}_r}{\partial \hat{r}} + 2\hat{H}_r + \frac{\partial \hat{H}_\theta}{\partial \theta} + \hat{H}_\theta \cot\theta = 0, \tag{5.15}$$

$$\beta\left(\hat{r}\frac{\partial \hat{E}_r}{\partial \hat{r}} + 2\hat{E}_r + \frac{\partial \hat{E}_\theta}{\partial \theta} + \hat{E}_\theta \cot\theta\right) = \hat{r}\frac{\partial \hat{H}_\theta}{\partial \hat{r}} + \hat{H}_\theta - \frac{\partial \hat{H}_r}{\partial \theta} \tag{5.16}$$

$$\frac{2\beta\hat{w}}{1+\beta^2} = \hat{E}_r\hat{H}_\theta - \hat{E}_\theta\hat{H}_r\,, \quad \hat{w} = \frac{w}{W_n} = \frac{1}{2}(\hat{E}_r^2 + \hat{E}_\theta^2 + \hat{H}_r^2 + \hat{H}_\theta^2)\,. \tag{5.17}$$

Similarly, we obtain from (5.5),

$$\mathbf{j}\,\hat{r}\hat{q} = \hat{r}\frac{\partial\hat{E}_r}{\partial\hat{r}} + 2\hat{E}_r + \frac{\partial\hat{E}_\theta}{\partial\theta} + \hat{E}_\theta\cot\theta\,. \tag{5.18}$$

All these equations involve only dimensionless quantities and are of pure mathematical nature in the sense that they are free of any physical parameter. Additional normalized quantities are introduced hereafter without further comment.

5.3 Major characteristic values of a rotating field

We start by defining the *total energy*, W, the *total classical mass*, M_c, and the *total charge*, Q:

$$W = \int_V w\,\mathrm{d}V = M_c c^2\,, \quad M_c = \frac{1}{c^2}\int_V m_c\,\mathrm{d}V\,, \quad \text{(a),(b)}\Big\} \tag{5.19}$$

$$Q = \int_V q\,\mathrm{d}V = r_n^2\sqrt{\varepsilon W_n}\,\hat{Q}, \quad \hat{Q} := \int_{\hat{V}}\hat{q}\,\mathrm{d}\hat{V}\,, \tag{5.20}$$

where, expressed in term of spherical coordinates, $\mathrm{d}V$ and $\mathrm{d}\hat{V}$ are given by

$$\mathrm{d}V = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi, \quad \mathrm{d}\hat{V} = \hat{r}^2\sin\theta\,\mathrm{d}\hat{r}\,\mathrm{d}\theta\,\mathrm{d}\varphi\,.$$

The integrations have, in principle, to be extended over the entire space spanned by the position coordinates r, θ, φ and $\hat{r}$, θ, φ, respectively.

In spherical coordinates the momentum density (cf. (4.3) and (5.12)) is given by

$$\mathbf{j} = (0,0,j)^{\mathrm{T}} = \frac{1}{c^2}\mathbf{S} = \frac{1}{c^2}(0,0,S)^{\mathrm{T}}, \quad j := m_i v\,. \tag{5.21}$$

On the other hand, the *angular momentum density* $\mathbf{l}$, defined with respect to the origin, thus to the centre of the field, is related to $\mathbf{j}$ by

$$\mathbf{l} = (l_x, l_y, l_z)^{\mathrm{T}} = \mathbf{r}\times\mathbf{j}, \quad \mathbf{r} = (x,y,z)^{\mathrm{T}},$$

where we have represented the vector $\mathbf{l}$ and the position vector $\mathbf{r}$ in terms of Cartesian components. We have,

$$l_x = jr\sin\theta = Sr\sin\theta/c^2\,.$$

Due to the circular symmetry of the field, the *total angular momentum*, $\mathbf{L} = (L_x, L_y, L_z)^{\mathrm{T}}$, is simply given by

$$\left.\begin{aligned} &\mathbf{L} = (L,0,0)^{\mathrm{T}}, \quad L = \int_V l_x \mathrm{d}V = \frac{1}{c} r_n^4 W_n \hat{L}, \\ &\hat{L} := \int_{\hat{V}} \hat{S}\hat{r}\sin\theta \,\mathrm{d}\hat{V}, \quad \hat{S} := \hat{E}_r \hat{H}_\theta - \hat{E}_\theta \hat{H}_r. \end{aligned}\right\} \tag{5.22}$$

The presence of the current density $\mathbf{i} = q\mathbf{v}$ causes the rotating field to exhibit also a *magnetic moment*, $\mathbf{L}_m$, which can be defined by [10],

$$\mathbf{L}_m = \frac{1}{2}\int_V (\mathbf{r}\times\mathbf{v}) q \,\mathrm{d}V .$$

Again due to the circular symmetry of the field, $\mathbf{L}_m$ is parallel to that axis and is given by

$$\mathbf{L}_m = (L_m,0,0)^{\mathrm{T}}, \quad L_m := \int_V l_m \mathrm{d}V, \quad l_m := \frac{1}{2} qvr\sin\theta; \tag{5.23}$$

it can be evaluated once the functions q and v are known.

5.4 Sommerfeld's fine structure constant

Consider now the quantity $\tilde{F}$ defined by

$$\tilde{F} := \frac{Q^2}{|L|}\sqrt{\frac{\mu}{\varepsilon}} = \frac{\hat{Q}^2}{|\hat{L}|} = \frac{\left(\int_{\hat{V}} \hat{q}\,\mathrm{d}\hat{V}\right)^2}{\left|\int_{\hat{V}} \hat{S}\hat{r}\sin\theta \,\mathrm{d}\hat{V}\right|}, \tag{5.24}$$

where the last two expressions are obtained from (5.20) and (5.22). The right-hand side in (5.24) can entirely be determined by means of (5.14) to (5.18). It is thus not only dimensionless but independent of any physical parameter, and the same is therefore true for $\tilde{F}$. Hence, $\tilde{F}$ is a pure mathematical number, in a sense like the number π, which can also be determined either as ratio of two quantities obtained by physical measurements, the circumference and the diameter of a circle, or by pure mathematical reasoning. Clearly, the most difficult step for finding $\tilde{F}$ is to find an acceptable solution of the underdetermined set of nonlinear PDEs specified by (5.14) to (5.17). Some requirements that have to be imposed on any approach attempting to solve (5.6) to (5.10) are outlined in Appendix D.

The problem itself appears to be of considerable interest. Consider indeed the so-called *fine-structure constant*, F, which is a dimensionless quantity defined by

$$F = \frac{Q^2}{2h\varepsilon c} = \frac{Q^2}{2h} Z, \quad Z = \sqrt{\frac{\mu}{\varepsilon}}, \tag{5.25}$$

where Q is the electron charge, Z the impedance of free space, and h the Planck constant [21, 24-26]. The values of the quantities on the right-hand side of (5.25) are known from experiments (and by related convention), and F can this way be determined to be quite close to 1/137. Furthermore, the spin of an electron, thus its angular momentum, is known to be given by $\hbar/2 = h/4\pi$. Hence, the analogy between F and $\tilde{F}$ is striking. Is this purely accidental or could it be a hint that the rotating EM field considered here is a valid electron model? If the latter were the case, the present approach would open a road to determining F by a purely mathematical process, as is the case for π. According to (5.24) and (5.25) we would then have $F = \tilde{F}/8\pi$. Furthermore, the quantities W and M_c introduced in Subsection 5.3 would be those known as rest energy and rest mass of the electron. Or else, if from the point of view of an outside observer the electron is not moving, W is its total energy and M_c its mass. Both these quantities do however depend on the internal field velocity of the rotating EM fluid and may not be confused with W_0 and M_0, from which they clearly differ, as will also be discussed in Subsection 5.6. The case of a rotating field that is moving as a whole will be addressed in Subsections 7.1 and 7.2.

5.5 Incompleteness of the original formulation

Let $\mathbf{E}$, $\mathbf{H}$, $\mathbf{i}$, and q be a solution of (2.1) and (2.2). If we then multiply these quantities by some arbitrary constant, say k, the equations (2.1) and (2.2) are still satisfied, but the values of $\mathbf{v}$ and $\boldsymbol{\beta}$ given by (2.11) remain unchanged. This observation is of particular interest for a basal field, as we are considering here. Thus, if $\mathbf{E}_r$, $\mathbf{E}_\theta$, $\mathbf{H}_r$, $\mathbf{H}_\theta$, and q are all multiplied by k while v is kept the same as before, (5.4) to (5.11) remain satisfied. Furthermore, as can easily be verified, $\hat{F}$ remains unaffected. This amounts to saying that one of the relevant quantities, say Q, is arbitrary, or else, that the statement of the present problem is incomplete.

This is an unavoidable consequence of our restricting the analysis to a problem exclusively specified by Maxwell's equations, as can be seen by

applying an elementary dimensional analysis. The only parameters involved are indeed ε and μ, and it is easily verified that the product $\varepsilon^{\nu_1}\mu^{\nu_2}$, where ν_1 and ν_2 (as well as ν_3 and ν_4 to be used hereafter) are real numbers, cannot possibly have the dimension of a pure charge. Hence, phenomena other than those of pure EM nature must play a role. Since according to Section 4, inertia, thus mass, is involved (cf. (4.31) and (4.32)), gravitation comes to mind as a candidate. This would add the gravitational constant G to the EM constants, but $\varepsilon^{\nu_1}\mu^{\nu_2}G^{\nu_3}$ cannot yield a pure charge either.

There thus remains an important open question: Which additional phenomenon could actually be responsible for fixing the value of Q, or equivalently that of L in (5.24) (or that of h in (5.25))? Purely formally, one can solve the dilemma by introducing a further physical constant, G_1, such that $\varepsilon^{\nu_1}\mu^{\nu_2}G^{\nu_3}G_1^{\nu_4}$ has the needed property. It can be verified that such an additional constant would exist if the gravitational law were to be complemented by a repulsive term that decreases sufficiently faster than $1/r^2$ and thus is irrelevant except at extremely small values of r. At present, however, such an observation is purely speculative.

5.6 Nominal values of frequency, velocity, and radius

It is instructive to consider, in addition to densities such as w_0, m_0, etc., also the corresponding total values

$$\left.\begin{aligned} W_0 &:= \int_V w_0 \mathrm{d}V, \quad M_0 := \int_V m_0 \mathrm{d}V, \\ W_i &:= \int_V w_i \mathrm{d}V, \quad M_i := \int_V m_i \mathrm{d}V, \\ W_{ke} &:= \int_V w_{ke} \mathrm{d}V, \quad W_{ki} := \int_V w_{ki} \mathrm{d}V, \\ W_k &:= \int_V w_k \mathrm{d}V = 2\int_V \frac{\beta^2}{\alpha^2} w_0 \mathrm{d}V, \\ &= 2W_{ke} = 2W_{ki}, \end{aligned}\right\} \tag{5.26}$$

where in (5.26) we have made use of (4.34). To the relations just listed should be added the characteristic values introduced in Subsection 5.3. Using (5.21), we do, however, rewrite L in the following form:

$$\left.\begin{aligned} L &= \int_V jR\,\mathrm{d}V = \int_V m_i v R\,\mathrm{d}V = \int_V \frac{v}{\alpha^2} m_0 R\,\mathrm{d}V, \\ R &:= r\sin\theta, \end{aligned}\right\} \tag{5.27}$$

where R is the cylindric (lateral) radius. Since the densities are not independent of one another (cf. (4.31) to (4.37)), the same holds for the corresponding total values. We thus have, for instance,

We next introduce for β, α, v, R, and ω corresponding *nominal* (mean) *values* $\overline{\beta}$, $\overline{\alpha}$, $\overline{v}$, $\overline{R}$, and $\overline{\omega}$, which, contrary to the former, are independent of r, θ, (and φ). We address them consecutively and consider first $\overline{\beta}$, thus also $\overline{\alpha}$ and $\overline{v}$. In terms of W_k and W_0, we define them by

$$\left.\begin{aligned} W_k &= 2\int_V \frac{\beta^2}{1-\beta^2} w_0\,\mathrm{d}V = 2\frac{\overline{\beta}^2}{1-\overline{\beta}^2}\int_V w_0\,\mathrm{d}V \\ &= 2\frac{\overline{\beta}^2}{1-\overline{\beta}^2} W_0 = 2\frac{\overline{\beta}^2}{\overline{\alpha}^2} W_0, \end{aligned}\right\} \tag{5.28}$$

$$\overline{\alpha}^2 = 1-\overline{\beta}^2, \quad \overline{v} = c\overline{\beta}. \tag{5.29}$$

We then define $\overline{R}$ in terms of L, M_0, and $\overline{\beta}$ by

$$L = \int_V \frac{v}{\alpha^2} R m_0\,\mathrm{d}V = \frac{\overline{v}}{\overline{\alpha}^2}\overline{R}\int_V m_0\,\mathrm{d}V = \frac{\overline{v}}{\overline{\alpha}^2}\overline{R}M_0, \tag{5.30}$$

and finally $\overline{\omega}$ by means of

$$\overline{\omega} = \overline{v}\,/\,\overline{R}\,. \tag{5.31}$$

As can be verified, this yields the following interesting expressions

$$\overline{\beta}^2 = \frac{W-W_0}{W+W_0}, \quad W_i = \frac{W_0}{\overline{\alpha}^2} = \frac{W}{1+\overline{\beta}^2},$$

and in particular,

$$W_k = \overline{\omega}L, \tag{5.32}$$

which will be of interest in Subsections 6.9 and 7.2, and

$$\overline{R} = \frac{1+\overline{\beta}^2}{2|\overline{\beta}|}R_0, \quad R_0 := \frac{c|L|}{W} = \frac{|L|}{cM_c}. \tag{5.33}$$

Alternatively, we could have started from (5.26) and (5.27), which could have led us directly to (5.32), but the procedure adopted here appears to

be more instructive, although both ways of proceeding have their merits. Clearly, L always has the same sign as $\overline{v}$ (and thus $\overline{\beta}$) and $\overline{\omega}$.

The interpretation of (5.28) and (5.30) in terms of the standard mean-value theorem of integration calculus hinges of course on the assumption that no unpermitted changes of sign occur in the integrand of the respective first integral. No problem can arise for (5.28). In the case of (5.30) it appears still plausible that v is either always non-negative or always non- positive, but there is no strict guarantee so far that this is indeed the case.

As follows from (5.33), if one assumes L and W given but $\overline{\beta}$ variable, R_0 is the minimum value that can be achieved by $\overline{R}$; this minimum is reached for $|\overline{\beta}|=1$. If in the definition of R_0 one sets $|L|=\hbar/2$ and W equal to the rest energy of an electron, one finds, restricting ourselves to 5 significant digits, $R_0 = 1.9308\times 10^{-13}$ meters. For comparison, let R_c be the *classical electron radius* and R_B the *Bohr radius.* We obtain $R_B / R_0 = 274.02$ and $R_0 / R_c = 68.518$. Hence, R_0 is substantially larger than the so-called classical electron radius but much smaller than the radius of, say, a hydrogen atom. We also have $2R_0 = \sqrt{R_B R_c}$, which, as follows from the defining expressions for R_0, R_B, and R_c, is in fact an exact relation. These results are quite plausible. For the product $\overline{\beta}\overline{R}$ we find

$$\frac{1}{2} R_0 < |\overline{\beta}|\,\overline{R} \le R_0 .$$

In a similar way, nominal values can also be obtained by starting from the integral expression (5.23), in which case a relation between the total magnetic moment L_m, the total charge Q, and the nominal value of $|\beta|R$ is obtained. However, the latter cannot be expected to be strictly the same as $|\overline{\beta}|\,\overline{R}$ defined above, but the two should differ at most by a moderate amount. We do not examine this here in more detail but want to add at least the following observation: While both $\mathbf{E}$ and $\mathbf{H}$ have the same kind of influence upon w, q and therefore i are essentially determined by $\mathbf{E}$ alone. At the centre of the field, $\mathbf{H}$ is high while $\mathbf{E}=\mathbf{0}$

(cf. Appendix D) and $q = 0$. Furthermore, for somewhat increased values of r and R, $\mathbf{H}$ is smaller and $\mathbf{E}$ larger; in particular, q is larger but m_i is much less affected, i.e., l_m becomes noticeably larger compared to l. Consequently, the ratio L_m / L must itself be noticeably larger than for a field configuration for which l_m / l (cf. (5.22)(a) and (5.23)) were constant. This is in agreement with what is known to be experimentally observed in the case of an electron.

6. Localized basal electromagnetic fields: planar field

6. 1 Determination of the general solution

We consider an autonomous EM field that is moving in the *x*-direction with field velocity $\mathbf{v} = (c, 0, 0)^{\mathrm{T}}$, in which case $\alpha = 0$ and $w_0 = 0$ (cf. Subsections 2.3 and 2.4). According to (2.15) and (2.34) we have

$$E_x = H_x = 0, \quad \varepsilon(E_y^2 + E_z^2) = \mu(H_y^2 + H_z^2),$$
$$E_y H_y + E_z H_z = 0.$$

The only two mathematical solutions the last two equations admit are $\sqrt{\varepsilon} E_y = \pm\sqrt{\mu} H_z$, $\sqrt{\varepsilon} E_z = \mp\sqrt{\mu} H_y$. For the Poynting vector they lead to

$$S_x = E_y H_z - E_z H_y = \pm(E_y^2 + E_z^2)\sqrt{\varepsilon / \mu}\,.$$

Since we must have $S_x > 0$, the only physically admissible solution is

$$E_x = H_x = 0, \quad \sqrt{\varepsilon} E_y = \sqrt{\mu} H_z, \quad \sqrt{\varepsilon} E_z = -\sqrt{\mu} H_y. \tag{6.1}$$

If we substitute (6.1) in (2.1) and (2.2) , we find as sole requirements

$$\left.\begin{aligned} &\frac{\partial E_y}{\partial t} + c\frac{\partial E_y}{\partial x} = 0, \quad \frac{\partial E_z}{\partial t} + c\frac{\partial E_z}{\partial x} = 0, \quad \text{(a),(b)} \\ &i_y = i_z = 0, \quad i_x = cq, \qquad \text{(c),(d),(e)} \end{aligned}\right\} \tag{6.2}$$

$$\left.\varepsilon\left(\frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = q, \quad \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0. \quad \text{(a),(b)}\right\} \tag{6.3}$$

From (6.2)(a) and (b) we conclude that E_y and E_z depend on x and t only via $x - ct$, or equivalently, via

$$\tau = t - \frac{x}{c}, \tag{6.4}$$

and the same can then be seen to hold also for q and thus for i_x; we may express this by writing

$$E_y = E_y(\tau, y, z), \quad E_z = E_z(\tau, y, z),$$
$$q = q(\tau, y, z), \qquad i_x = i_x(\tau, y, z).$$

As for the equations (6.2) (c) to (d), they amount to stating that $\mathbf{i} = q\mathbf{v} = q\mathbf{v}_{ch}$ and therefore show that the field is basal in the given reference frame (cf. the first full sentence following (2.45)). On the other hand, (6.3) is known to be equivalent to the existence of a function (negative potential) ψ that depends on y and z (and, of course, on τ) and is such that

$$\left.\begin{aligned} &\frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{\varepsilon} q, && \text{(a)} \\ &E_y = \frac{\partial\psi}{\partial y}, \quad E_z = \frac{\partial\psi}{\partial z}, && \text{(b)} \end{aligned}\right\} \tag{6.5}$$

where (6.5)(a) is the two-dimensional Poisson equation. For E_y and E_z vanishing at infinity its solution is

$$\left.\begin{aligned} &\psi = \frac{1}{2\pi\varepsilon} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q(\tau, \hat{y}, \hat{z}) \ln \hat{d} \cdot \mathrm{d}\hat{y}\,\mathrm{d}\hat{z}, && \text{(a)} \\ &\hat{d} := \sqrt{(y-\hat{y})^2 + (z-\hat{z})^2}, \quad \tau = t - \frac{x}{c}, && \text{(b)} \end{aligned}\right\} \tag{6.6}$$

and therefore, the differentiations under the integral in (6.6)(a) being permitted [27],

$$\left.\begin{aligned} E_y &= \frac{1}{2\pi\varepsilon} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(y-\hat{y})\, q(\tau, \hat{y}, \hat{z})}{\hat{d}^2} \mathrm{d}\hat{y}\,\mathrm{d}\hat{z}, \\ E_z &= \frac{1}{2\pi\varepsilon} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(z-\hat{z})\, q(\tau, \hat{y}, \hat{z})}{\hat{d}^2} \mathrm{d}\hat{y}\,\mathrm{d}\hat{z}, \end{aligned}\right\} \tag{6.7}$$

where $\hat{d}$ is the distance between the points (y, z) and $(\hat{y}, \hat{z})$ for x and t held fixed, thus for τ (cf. also (6.4)) constant.

6.2 Spectral properties

Let us look more closely at a solution that occupies, essentially, a narrow spectral band extending, for (angular) frequencies $\omega > 0$, from $\Omega - \Delta\omega/2$ to $\Omega + \Delta\omega/2$, where $\Delta\omega$ is the *bandwidth*, Ω the *nominal frequency*, and $\Omega >> \Delta\omega > 0$. The choice of Ω cannot rigorously be defined; it should be a representative value inside of the relevant frequency range and may be thought of as its centre frequency. Concerning their dependence on t, the functions E_y, E_z, H_y, H_z, q, and i_x are then of the same type as signals in communications, in fact of a generalized, combined amplitude- and phase-modulated type with (suppressed) carrier of frequency equal to Ω [28-30]. We assume Ω to be independent of x, y, z, and t.

For ease of terminology, we will hereafter often speak about signals when referring to E_y, E_z, H_y, H_z, q, and i_x. Since the situation is the same for all those signals, we represent them generically by f and use a corresponding notation for all related functions of interest. In view of what we have seen above, f may be defined as a function of τ (cf. (6.4)), y, and z. We may express this by writing $f = f(\tau, y, z)$.

Let $\mathcal{F}_t\{f\}$ be the Fourier transform of f with respect to t (cf. (E.1)(a)) and $F = \mathcal{F}_\tau\{f\}$ the corresponding transform with respect to τ, i.e.

$$F(j\omega, y, z) = \int_{-\infty}^{\infty} f(\tau, y, z) e^{-j\omega\tau} \mathrm{d}\tau \ . \tag{6.8}$$

Due to (6.4), the two transforms are simply related by

$$\mathcal{F}_t\{f(\tau, y, z)\} = e^{-j\omega x/c} F(j\omega, y, z) \, .$$

We may therefore concentrate on F alone.

Since the signals f we are interested in are real and the equations we have to consider are linear, it is helpful to make use of suitable complex representations, preferably by appealing to both the *analytic signal* f_+ and the *complex amplitude* $\hat{f}$ (in the literature sometimes referred to, although somewhat inadequately, as complex envelope) associated with f, as explained in Appendix E1. Both f_+ and $\hat{f}$ are indeed complex-valued functions of real variables.

The analytic signal $f_+ = f_+(\tau, y, z)$ can easiest be defined by means of its Fourier transform $F_+ = \mathcal{F}_\tau\{f_+\}$, i.e. (cf. (E.2)), by

$$F_+(j\omega, y, z) = \begin{cases} 2F(j\omega, y, z) & \text{for } \omega > 0, \\ 0 & \text{for } \omega < 0, \end{cases} \tag{6.9}$$

and we have (cf. (E.3)(a)),

$$f(\tau, y, z) = \operatorname{Re} f_+(\tau, y, z). \tag{6.10}$$

As is known, $\operatorname{Im} f_+$, thus the imaginary part of f_+, is the Hilbert transform of f. Identifying ω_0 in (E.5) with the present Ω, the *complex amplitude* $\hat{f}$ that corresponds to f is defined by

$$f_+(\tau, y, z) = e^{j\Omega\tau}\hat{f}(\tau, y, z), \tag{6.11}$$

which in turn (cf. (E.5)) is equivalent to

$$\hat{F}(j\omega, y, z) := \mathcal{F}_\tau\{\hat{f}\} = F_+(j\omega + j\Omega, y, z). \tag{6.12}$$

Clearly, the above-mentioned narrow-band assumption refers directly to properties of $\mathcal{F}_\tau\{f\}$. For reasons of simplicity we assume for the present purpose that the band limitation is strict, i.e., that it can be expressed by stating

$$\left.\begin{aligned} &F_+(j\omega, y, z) = 0 \quad \text{for } \omega \notin I \quad \text{and } \forall y, z, && \text{(a)} \\ &I := \{\omega \,|\, \Omega - \frac{1}{2}\Delta\omega \le \omega \le \Omega + \frac{1}{2}\Delta\omega\}, \quad 0 < \Delta\omega \ll \Omega, && \text{(b)} \end{aligned}\right\} \tag{6.13}$$

or, due to (6.12),

$$\hat{F}(j\omega, y, z) = 0 \quad \text{for} \quad |\omega| > \frac{\Delta\omega}{2}. \tag{6.14}$$

Hence, the complex amplitude $\hat{f}(\tau, y, z)$ occupies only a narrow spectrum, which extends from $-\Delta\omega/2$ to $\Delta\omega/2$, and is therefore slowly varying in τ compared to $e^{j\Omega\tau}$.

6.3 Change of reference frame

Consider again a second reference frame RF′ as, say, in Subsection 2.1. We restrict ourselves to assuming that the direction of $\mathbf{v}_0$ coincides with the one in which the field is propagating, i.e., that (2.3) holds. We may thus immediately combine (2.3) to (2.7) with (6.1) and (6.2)(b). This can be verified to yield,

$$\Omega\tau = \Omega'\tau', \quad y = y', \quad z = z', \quad \text{(a), (b), (c)}\}, \tag{6.15}$$

$$E_x = E'_x = H_x = H'_x = i_y = i'_y = i_z = i'_z = 0,$$

$$\frac{E'_y}{E_y} = \frac{E'_z}{E_z} = \frac{H'_y}{H_y} = \frac{H'_z}{H_z} = \frac{q'}{q} = \frac{i'_x}{i_x} = \sqrt{\frac{1-\beta_0}{1+\beta_0}}, \tag{6.16}$$

where τ' and Ω' are defined by

$$\tau' = t' - \frac{x'}{c}, \quad \Omega' = \Omega\sqrt{\frac{1-\beta_0}{1+\beta_0}}. \quad \text{(a), (b)}\Big\}. \tag{6.17}$$

Since according to (6.16) the transformation from unprimed to primed quantities is the same for all six variables E'_y, E'_z, H'_y, H'_z, q', and i'_x, we may again represent all these signals generically by a same symbol, say f', and use again a corresponding notation for all related functions of interest. In particular, (6.16) can be expressed in the form,

$$\frac{f'(\tau', y', z')}{\Omega'} = \frac{f(\tau, y, z)}{\Omega} = \frac{1}{\Omega} f(\tau' \frac{\Omega'}{\Omega}, y, z), \tag{6.18}$$

where the second equality follows simply by making use of (6.15)(a).

Since the expression for τ' is of exactly same type as that for τ (cf. (6.4) and (6.17)(a)), all equations in Subsection 6.2 that are of present interest, in particular those up to (6.12), apply also after adding primes to all variable quantities. Accordingly, we conclude, first from (6.18),

$$F'(j\omega', y', z') := \mathcal{F}_{\tau'}\{f'\} = \int_{-\infty}^{\infty} f'(\tau', y', z') e^{-j\omega'\tau'} \mathrm{d}\tau'$$

$$= \frac{\Omega'}{\Omega} \int_{-\infty}^{\infty} f(\tau' \frac{\Omega'}{\Omega}, y, z) e^{-j\omega'\tau'} \mathrm{d}\tau',$$

and then from (6.8) and (6.15) (a), since the ratio Ω' / Ω is constant,

$$F'(j\omega', y', z') = \int_{-\infty}^{\infty} f(\tau, y, z) e^{-j\omega'\Omega\tau/\Omega'} \mathrm{d}\tau$$

$$= F(j\omega' \frac{\Omega}{\Omega'}, y, z). \tag{6.19}$$

We define F'_+ analogously to F_+ in (6.9) and thus obtain first from (6.19),

$$F'_+(j\omega', y', z') = \begin{cases} 2F'(j\omega', y', z') = 2F(j\omega' \dfrac{\Omega}{\Omega'}, y, z) & \text{for } \omega' > 0, \\ 0 \quad \text{for } \omega' < 0, \end{cases}$$

and then by using (6.9),

$$\left.\begin{aligned} &F'_+(j\omega', y', z') = F_+(j\omega, y, z) \quad \forall\omega', \quad \text{(a)} \\ &\text{where} \quad \omega = \omega'\frac{\Omega}{\Omega'}. \qquad\qquad \text{(b)} \end{aligned}\right\} \tag{6.20}$$

According to (6.13) the right-hand side in (6.20)(a) vanishes for all $\omega \notin I$, and this irrespective of x, y, and z. Due to (6.20)(b), the left-hand side of (6.20)(a) thus vanishes for all $(\omega'\Omega/\Omega') \notin I$, or equivalently,

$$F'_+(j\omega', y', z') = 0 \quad \text{for } \omega' \notin I' \quad \text{and} \quad \forall y', z',$$

where I' and $\Delta\omega'$ are defined by (6.21)(a) and (b),

$$\left.\begin{aligned} &I' = \{\omega' \,|\, \Omega' - \frac{1}{2}\Delta\omega' \le \omega' \le \Omega' + \frac{1}{2}\Delta\omega'\}, \quad \text{(a)} \\ &\frac{\Delta\omega'}{\Omega'} = \frac{\Delta\omega}{\Omega}, \quad 0 < \Delta\omega' \le \Omega', \qquad \text{(b), (c)} \end{aligned}\right\} \tag{6.21}$$

and where the property (6.21)(c) follows from the corresponding one in (6.13)(b). This confirms that ω', Ω', and $\Delta\omega'$ play indeed the same role in RF′ as ω, Ω, and $\Delta\omega$ do in RF. In particular, the relative bandwidth $\Delta\omega/\Omega$ is independent of the adopted reference frame, while passing from Ω to Ω' and from ω to ω' clearly amounts to what is known as the (longitudinal) *Doppler effect*.

6.4 Charge and polarization

6.4.1 Linear polarization

We obviously still have large freedom for choosing q. In contrast to the rotating field considered in Section 5, we want the present field to be *electrically neutral*, i.e. such that its *effective* (total) charge is always zero. Furthermore, we would like to achieve an appropriate polarization of the electric field. At present (see however Subsection 6.4.2), we assume this to be an *effective* linear polarization, say in the *y*-direction. An appropriate way to achieve both is to assume q to be distributed symmetrically with respect to the (*x*, *y*)-plane and anti-symmetrically with respect to the (*x*, *z*)-plane, thus to require, with τ given by (6.4),

$$q(\tau, y, -z) = q(\tau, y, z), \quad q(\tau, -y, z) = -q(\tau, y, z). \tag{6.22}$$

In view of (6.6), (6.22) leads to

$$\left.\begin{aligned} E_y(\tau,y,z) &= \frac{1}{2\pi\varepsilon}\int_0^\infty\int_0^\infty A_y(y,z,\hat{y},\hat{z})\,q(\tau,\hat{y},\hat{z})\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{z},\\ E_z(\tau,y,z,) &= \frac{1}{2\pi\varepsilon}\int_0^\infty\int_0^\infty A_z(y,z,\hat{y},\hat{z})\,q(\tau,\hat{y},\hat{z})\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{z},\end{aligned}\right\} \tag{6.23}$$

where A_y and A_z are defined by

$$\begin{aligned} A_y(y,z,\hat{y},\hat{z}) = {} & \frac{y-\hat{y}}{(y-\hat{y})^2+(z-\hat{z})^2} - \frac{y+\hat{y}}{(y+\hat{y})^2+(z-\hat{z})^2}\\ & + \frac{y-\hat{y}}{(y-\hat{y})^2+(z+\hat{z})^2} - \frac{y+\hat{y}}{(y+\hat{y})^2+(z+\hat{z})^2}, \end{aligned} \tag{6.24}$$

$$\begin{aligned} A_z(y,z,\hat{y},\hat{z}) = {} & \frac{z-\hat{z}}{(y-\hat{y})^2+(z-\hat{z})^2} - \frac{z+\hat{z}}{(y-\hat{y})^2+(z+\hat{z})^2}\\ & - \frac{z-\hat{z}}{(y+\hat{y})^2+(z-\hat{z})^2} + \frac{z+\hat{z}}{(y+\hat{y})^2+(z+\hat{z})^2}. \end{aligned} \tag{6.25}$$

From (6.23) to (6.25) we derive the simple relations

$$E_y(\tau,-y,z,) = E_y(\tau,y,z), \quad E_y(\tau,y,-z) = E_y(\tau,y,z), \tag{6.26}$$

$$E_z(\tau,-y,z,) = -E_z(\tau,y,z), \quad E_z(\tau,y,-z) = E_z(\tau,y,z). \tag{6.27}$$

Let us then define *effective* field components by means of

$$\left.\begin{aligned} E_{y\,eff}(\tau) &= \int_{-\infty}^\infty\int_{-\infty}^\infty E_y(\tau,y,z)\,\mathrm{d}y\,\mathrm{d}z,\\ E_{z\,eff}(\tau) &= \int_{-\infty}^\infty\int_{-\infty}^\infty E_z(\tau,y,z)\,\mathrm{d}y\,\mathrm{d}z.\end{aligned}\right\} \tag{6.28}$$

Using (6.26) and (6.27) we find

$$\left.\begin{aligned} E_{y\,eff}(\tau) &= 4\int_0^\infty\int_0^\infty E_y(\tau,y,z)\,\mathrm{d}y\,\mathrm{d}z, \quad \text{(a)}\\ E_{z\,eff}(\tau) &= 0 \quad \forall\tau. \qquad\qquad\quad \text{(b)}\end{aligned}\right\} \tag{6.29}$$

Hence, the field has an *effective polarization* in the y-direction.

The effective field components can also be expressed in terms of the associated complex amplitudes $\hat{E}_{y\,eff}$ and $\hat{E}_{z\,eff}$. We have indeed (cf. (E.3)(a) and (E.5)(a)),

$$E_{y\,eff}(\tau) = \mathrm{Re}\{\hat{E}_{y\,eff}(\tau)e^{j\Omega\tau}\}, \quad E_{z\,eff}(\tau) = \mathrm{Re}\{\hat{E}_{z\,eff}(\tau)e^{j\Omega\tau}\}, \quad \text{(a), (b)}\Big\} \tag{6.30}$$

where, $E_{y\,eff+}$ and $E_{z\,eff+}$ being the analytic signals corresponding to $E_{y\,eff}$ and $E_{z\,eff}$,

$$\hat{E}_{y\,eff}(\tau) = E_{y\,eff+}(\tau)e^{-j\Omega\tau}, \quad \hat{E}_{z\,eff}(\tau) = E_{z\,eff+}(\tau)e^{-j\Omega\tau}.$$

More specifically, (6.29)(b)) implies

$$\hat{E}_{z\,eff}(\tau) = E_{z\,eff+}(\tau) = 0 \quad \forall\tau. \tag{6.31}$$

Clearly, although $\hat{E}_{y\,eff}$ is not simply a constant, as would be the case if we were dealing with monochromatic signals, it is slowly varying in τ time compared to $e^{j\Omega\tau}$.

As can be verified by means of (6.23) to (6.25), the equations (6.22), in particular thus the property of being electrically neutral, also have the important consequence that both A_y and A_z, and therefore both E_y and E_z vanish for $y \to \infty$ and $z \to \infty$ as $1/y^2$ and $1/z^2$, respectively. Hence, the integrals in

$$\left.\begin{aligned} W &= \int_V w\,\mathrm{d}V = \frac{1}{2}\int_V (\varepsilon E^2 + \mu H^2)\,\mathrm{d}V \\ &= \varepsilon\int_V (E_y^2 + E_z^2)\,\mathrm{d}V, \quad \mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \end{aligned}\right\} \tag{6.32}$$

where W is the *total energy* of the field, are convergent, as is obviously required.

6.4.2 Circular polarization

From the above-derived solution for a planar localized EM field with linear polarization a corresponding solution with circular polarization can be achieved by following essentially a known procedure. More care must be taken for certain details, however, since we are not simply dealing with a monochromatic signal but with one occupying a frequency range with non-vanishing bandwidth.

Consider indeed two linearly polarized fields that we assume to be both of exactly same type except that their directions of polarization are orthogonal to each other and that one of them is lagging the other by $T/4 = \pi/2\Omega$. We characterize these two fields by underlining once and twice all relevant quantities, and we still make use of τ as defined by (6.4). Let the simply underlined field, which is composed of

$\underline{E}_y = \underline{E}_y(\tau, y, z)$ and $\underline{E}_z = \underline{E}_z(\tau, y, z)$, be identical to the one described in Subsection 6.4.1. The doubly underlined field can then be obtained from the simply underlined one by means of

$$\left.\begin{aligned} \underline{\underline{E}}_y(\tau, y, z) &= -\underline{E}_z(\tau - \frac{T}{4}, -z, y), \\ \underline{\underline{E}}_z(\tau, y, z) &= \underline{E}_y(\tau - \frac{T}{4}, -z, y). \end{aligned}\right\} \tag{6.33}$$

Combined fields may then be defined by

$$\sqrt{2}E_y = \underline{E}_y + \underline{\underline{E}}_y, \quad \sqrt{2}E_z = \underline{E}_z + \underline{\underline{E}}_z, \tag{6.34}$$

as well as by

$$\sqrt{2}E_y = \underline{E}_y - \underline{\underline{E}}_y, \quad \sqrt{2}E_z = \underline{E}_z - \underline{\underline{E}}_z. \tag{6.35}$$

We first consider (6.34). Due to (6.33) we can write

$$\left.\begin{aligned} \sqrt{2}E_y(\tau, y, z) &= \underline{E}_y(\tau, y, z) - \underline{E}_z(\tau - \frac{T}{4}, -z, y), \\ \sqrt{2}E_z(\tau, y, z) &= \underline{E}_z(\tau, y, z) + \underline{E}_y(\tau - \frac{T}{4}, -z, y). \end{aligned}\right\} \tag{6.36}$$

According to (6.28) and (6.29)(b), which had indeed been obtained for linear polarization, we have

$$\begin{aligned} \underline{E}_{y\,eff}(\tau) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underline{E}_y(\tau, y, z)\,\mathrm{d}y\,\mathrm{d}z \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underline{E}_y(\tau, -z, y)\,\mathrm{d}y\,\mathrm{d}z, \end{aligned} \tag{6.37}$$

$$\begin{aligned} \underline{E}_{z\,eff}(\tau) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underline{E}_z(\tau, y, z)\,\mathrm{d}y\,\mathrm{d}z \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underline{E}_z(\tau, -z, y)\,\mathrm{d}y\,\mathrm{d}z = 0 \quad \forall\tau, \end{aligned} \tag{6.38}$$

where in both (6.37) and (6.38) the second integrals follow from the first by substituting $y \to -z$ and $z \to y$. Let us then apply the definitions (6.28) to E_y and E_z given by (6.36) and make use of (6.37) and (6.38), in particular thus of $\underline{E}_{z\,eff}(\tau) = 0 \ \ \forall\tau$. The result can be written in the form

$$\sqrt{2}E_{y\,eff}(\tau) = \underline{E}_{y\,eff}(\tau), \quad \sqrt{2}E_{z\,eff}(\tau) = \underline{E}_{z\,eff}(\tau - \frac{T}{4}). \tag{6.39}$$

It is instructive to express these results by making use of the complex amplitudes associated with the relevant quantities. For doing this let us

first replace in (6.36) to (6.39) the quantities E_y, $\underline{E}_y$, $E_{y\,eff}$, $\underline{E}_{y\,eff}$ E_z, $\underline{E}_z$, $E_{z\,eff}$, and $\underline{E}_{z\,eff}$ by their associated analytic signals, i.e. by E_{y+}, $\underline{E}_{y+}$, $E_{y\,eff+}$, $\underline{E}_{y\,eff+}$, E_{z+}, $\underline{E}_{z+}$, $E_{z\,eff+}$, and $\underline{E}_{z\,eff+}$, respectively, which is obviously permitted in view of the linear nature of (6.8) and (6.9) as well as (6.36) to (6.39). In particular, (6.39) then yields

$$E_{y\,eff+}(\tau) = E_+(\tau), \quad E_{z\,eff+}(\tau) = E_+(\tau - \frac{T}{4}), \tag{6.40}$$

where E_+ and the subsequently used $\hat{E}$ are defined by

$$E_+(\tau) = e^{j\Omega\tau}\hat{E}(\tau) = \frac{1}{\sqrt{2}}\underline{E}_{y\,eff+}(\tau). \tag{6.41}$$

In view of (6.10) and (6.11) we then deduce from (6.40),

$$\left.\begin{aligned} E_{y\,eff}(\tau) &= \mathrm{Re}\{e^{j\Omega\tau}\hat{E}(\tau)\}, \\ E_{z\,eff}(\tau) &= \mathrm{Re}\{e^{j\Omega(\tau - T/4)}\hat{E}(\tau - \frac{T}{4})\} = \mathrm{Im}\{e^{j\Omega\tau}\hat{E}(\tau - \frac{T}{4})\}, \end{aligned}\right\} \tag{6.42}$$

where we have taken into account $e^{-j\Omega T/4} = -j$. While (6.42) has been obtained by defining E_y and E_z according to (6.34), we can equally well adopt the definitions (6.35). If we then proceed in the same way as above we arrive at the result

$$E_{y\,eff}(\tau) = \mathrm{Re}\{e^{-j\Omega\tau}\hat{E}^*(\tau)\}, \quad E_{z\,eff}(\tau) = \mathrm{Im}\{e^{-j\Omega\tau}\hat{E}^*(\tau - \frac{T}{4})\}, \tag{6.43}$$

where $\hat{E}$ is as defined by (6.41) and where the asterisk denotes complex conjugation.

Again, like for linear polarization, $\hat{E}(\tau)$ in (6.42) and (6.43) is slowly varying in τ compared to $e^{j\Omega\tau}$. In particular, $\hat{E}(\tau - T/4)$ differs little from $\hat{E}(\tau)$. Hence, (6.42) and thus (6.34) describe a right-circularly polarized field, and (6.43), thus (6.35), a field that is left-circularly polarized. The factor $\sqrt{2}$ has been included in (6.34) and (6.35) for power/energy reasons etc. The convergence property mentioned at the end of Subsection 6.4.1 holds of course also for circularly polarized fields.

6.5 Nominal frequency, size, energy, and momentum

As (6.17)(b) shows, the available range $0 \le |\beta_0| < 1$ allows us in particular to convert any finite $\Omega > 0$ into any finite $\Omega' > 0$ and vice versa. Hence, the specific frequency that is observed is simply a matter of the adopted reference system. It is then instructive to replace the last term in (6.16) by Ω'/Ω and to write the result in form of the following six individual equations:

$$\left.\begin{aligned} &\frac{E_y}{\Omega} = \frac{E'_y}{\Omega'}, \quad \frac{H_y}{\Omega} = \frac{H'_y}{\Omega'}, \quad \frac{q}{\Omega} = \frac{q'}{\Omega'}, \quad \text{(a),(b), (c)} \\ &\frac{E_z}{\Omega} = \frac{E'_z}{\Omega'}, \quad \frac{H_z}{\Omega} = \frac{H'_z}{\Omega'}, \quad \frac{i_x}{\Omega} = \frac{i'_x}{\Omega'}, \quad \text{(d), (e),(f)} \end{aligned}\right\} \tag{6.44}$$

where the unprimed numerators depend only on τ, y, z and the primed ones only on τ', y', z'. We may therefore conclude from (6.15) that the ratios in the left-hand sides of the six equalities (6.44) are independent of the choice of RF provided we evaluate them at the same values of $\tau\Omega$, y, and z. In other words, making use of the period $T = 2\pi/\Omega$, the wavelength $\Lambda = cT = 2\pi c/\Omega$, and the relations $\tau\Omega = 2\pi(t - x/c)/T = -2\pi(x - ct)/\Lambda$ we may state that the field is: 1. stretched out proportionally to Λ in the forward direction; 2. spread out independently of Ω in the lateral directions; 3. stretched out proportionally to T in the direction of time.

Of particular interest is the comparison of the total field energy W in RF to that in RF′, i.e., to W'. Since (6.32) is valid independently of the type of polarization, the same holds for W', which obviously is given by

$$\left.\begin{aligned} W' = \frac{1}{2}\int_V (\varepsilon E'^2 + \mu H'^2)\,\mathrm{d}V' = \varepsilon \int_V (E'^2_y + E'^2_z)\,\mathrm{d}V', \\ \mathrm{d}V' = \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'. \end{aligned}\right\} \tag{6.45}$$

Since $\mathrm{d}x$ and $\mathrm{d}x'$ are defined for t and t' held fixed, respectively, we have $\mathrm{d}x = -c\,\mathrm{d}\tau$ and $\mathrm{d}x' = -c\,\mathrm{d}\tau'$ (cf. (6.4) and (6.17)(a)). Furthermore, since β_0 is a constant, it follows from (6.17)(b) that Ω' is independent of τ' (thus of t' and x'), y' and z'. In view of (6.15), we may thus write,

$$\Omega'\mathrm{d}x' = \Omega\mathrm{d}x, \quad \mathrm{d}y' = \mathrm{d}y, \quad \mathrm{d}z' = \mathrm{d}z, \tag{6.46}$$

altogether therefore $\Omega' \mathrm{d}V' = \Omega \mathrm{d}V$. Hence, we conclude from (6.32), (6.44)(a) and (d), (6.45), and (6.46) that $W'/\Omega' = W/\Omega$. In other words, the ratio W/Ω is independent of the reference frame adopted. This is in full agreement with the three statements at the end of the previous paragraph.

Let us now look separately at the low-frequency and the high-frequency constituents of E_y and E_z, i.e., at their complex amplitudes $\hat{E}_y$ and $\hat{E}_z$, and at their behaviour in terms of Ω. In particular, defining the phases χ_y and χ_z by

$$\hat{E}_y = |\hat{E}_y| e^{j\chi_y}, \quad \hat{E}_z = |\hat{E}_z| e^{j\chi_z},$$

we derive from (6.10), (6.11), and (6.32),

$$\begin{aligned} W = \frac{\varepsilon}{2} \int_V \Big(& |\hat{E}_y(\tau, y, z)|^2 \left[1 + \cos\left(2\Omega\tau + 2\chi_y(\tau, y, z) \right) \right] \\ & + |\hat{E}_z(\tau, y, z)|^2 \left[1 + \cos\left(2\Omega\tau + 2\chi_z(\tau, y, z) \right) \right] \Big) \mathrm{d}V, \end{aligned} \tag{6.47}$$

where the integrand depends on x as determined by (6.4). Due to the narrow-band assumption (cf. (6.13) and (6.14)), a change in x causes the cosine terms to oscillate fast compared to the changes the complex amplitudes are undergoing. Hence, the contributions due to the cosine terms may be assumed to cancel out. Furthermore, (6.47) is in fact independent of t while x may be replaced by $-x$. The result (6.47) is therefore equivalent to

$$W = \frac{\varepsilon}{2} \int_V \left(|\hat{E}_y(\frac{x}{c}, y, z)|^2 + |\hat{E}_z(\frac{x}{c}, y, z)|^2 \right) \mathrm{d}V. \tag{6.48}$$

Consider next the ratio $\tilde{\hbar} = W/\Omega$, for which we obtain from (6.48),

$$\left.\begin{aligned} & W = \tilde{\hbar}\Omega, && \text{(a)} \\ & \tilde{\hbar} = \frac{\varepsilon}{2\Omega} \int_V \left(|\hat{E}_y(\frac{x}{c}, y, z)|^2 + |\hat{E}_z(\frac{x}{c}, y, z)|^2 \right) \mathrm{d}V. && \text{(b)} \end{aligned}\right\} \tag{6.49}$$

As is clear from the above discussion, $\tilde{\hbar}$ is independent of the specific reference system in which the field is observed. It is also found to be independent not only of Ω but also of the low-frequency quantities χ_y

and χ_z, and thus of any changes these quantities may undergo due to the freedom still available for q.

Since $v = (c,0,0)^{\mathrm{T}}$ and thus $\beta = 1$, the momentum density of the field is given by $\mathbf{j} = (j,0,0)^{\mathrm{T}}$, where, due to (4.38), $j = w/c$. For the total momentum we obtain, making also use of (6.49))(a),

$$J = \int_V j \,\mathrm{d}V = \frac{W}{c} = \tilde{\hbar}\frac{\Omega}{c} \quad . \tag{6.50}$$

As we have seen in Subsection 2.3 and as is confirmed by applying (6.1) to (2.28)(b), the rest energy density w_0 vanishes everywhere, and we thus have for the total rest energy

$$W_0 = \int_V w_0 \mathrm{d}V = 0 \,.$$

Hence, defining the total internal and external kinetic energies W_{ki} and W_{ke} by

$$W_{ki} = \int_V w_{ki} \mathrm{d}V, \quad \text{and} \quad W_{ke} = \int_V w_{ke} \mathrm{d}V \,,$$

respectively, we have, due to (4.34), $W_{ki} = W_{ke}$ and therefore

$$W = W_0 + W_{ki} + W_{ke} = W_k, \quad W = W_k = \tilde{\hbar}\Omega = 2W_{ke} = 2W_{ki}, \tag{6.51}$$

where $W_k = W_{ke} + W_{ki}$ is the total kinetic energy.

We may think of $\tilde{\hbar}$ in (6.49) as having been defined for circular polarization. If we then replace E_y and E_z by (6.34) or (6.35) and proceed as we have done above for arriving at (6.48) via (6.47), we may conclude that in the case of linear polarization (6.49) does not only hold again but, due to the factor $\sqrt{2}$ in the adopted definition, the resulting $\tilde{\hbar}$ is, within the narrow-band approximation, the same as for circular polarization.

6.6 Uncertainty properties

In order to obtain an appropriate uncertainty relation we start from a well-known result in the theory of the Fourier integral. More precisely, we use an extension of the more commonly encountered result to the case of an analytic signal and to the related concept of a complex amplitude. As explained in Appendix E2, we may thus introduce an appropriate definition of the *duration* Δt (spread in time) and of the *bandwidth*

$\Delta\omega$ (spread in frequency) of a signal. Such definitions of Δt and $\Delta\omega$ are not strictly the same as those adopted in (6.18) and (6.19), but they have the known advantage of remaining meaningful even if no sharp limits for the relevant ranges can be given, and of lending themselves to convenient mathematical analysis. In particular, we can write (cf. (E.15)),

$$\Delta t \cdot \Delta\omega \geq 2 . \tag{6.52}$$

To Δt corresponds a *spread in position*

$$\Delta x = c\Delta t ,$$

and in view of (6.49)(a), $\Delta\omega$ corresponds to a *spread in energy*

$$\Delta W = \tilde{\hbar}\Delta\omega$$

and thus, in view of (6.50), to a *spread in momentum*

$$\Delta J = \frac{\Delta W}{c} = \frac{\tilde{\hbar}}{c}\Delta\omega .$$

Hence (6.52) is equivalent to

$$\left(\frac{\Delta x}{2}\right)\left(\frac{\Delta J}{2}\right) = \frac{1}{4}\tilde{\hbar}\Delta t \cdot \Delta\omega \geq \frac{1}{2}\tilde{\hbar} , \tag{6.53}$$

where $\Delta x / 2$ and $\Delta J / 2$ are the root mean square (RMS) deviations of x and J.

In view of these various results one is obviously tempted to set $\tilde{\hbar} = \hbar =$ Planck's constant. The localized field under consideration then exhibits classical photon properties. Furthermore, in view of the presence of fluctuating positive and negative charges, which are balancing each other in the field, it is not surprising that a photon can split into a pair of an electron and a positron (instead of, for instance, two photons of lower energy) once the energy exceeds a certain threshold. On the other hand, it is likely that the field configuration would also become unstable if Ω is getting too small, thus Λ too large, or else, if the length of the field configuration becomes too large compared to its lateral extension and to the size of the other relevant physical objects involved. Furthermore, a solution for $\Omega = 0$ is not feasible since according to (6.17)(b) (where Ω' assumes the role of the present Ω) this would require $\beta_0 = 1$, which has explicitly been excluded (cf. beginning of Subsection 6.5).

To be precise, (6.52) and (6.53) do not express a true uncertainty but rather give, in a sense, a lower bound for the degree of non-compactness of the field. If one accepts that nature follows some principle of economy

(due to which for instance stability can be achieved) one may suspect that an actual photon is indeed such that an appropriate bound is reached. This would imply that the dependence of any field component on x and thus on t could be of the form of one of the functions discussed in Appendix E2. The most general such function is that given by (E.23). Identifying thus Ω with ω_0 and assuming the field to be centred at $\tau = 0$, we may replace $t - t_0$ in (E.23) by $\tau = t - x/c$. For fixed values of y and z the dependence on x and t should thus be of the form

$$|C| e^{-A\tau} \cos\left(\Omega\tau - B(\tau) - \gamma\right).$$

The dependence on y and z could then also be similar to a Gauss function, thus to a function somewhat of the form $e^{-(y^2+z^2)/r_0^2}$, where r_0 is also a constant. If that is the case we obtain altogether for f (cf. Subsection 6.2) an expression of the form

$$\left.\begin{aligned} f &= f_0 e^{-A(\tau)-(y^2+z^2)/r_0} \cos\varphi(\tau), \\ \varphi(\tau) &= \Omega\tau - B(\tau) - \gamma, \end{aligned}\right\} \tag{6.54}$$

where f_0 is a positive constant. In a sense, the classical concept of a photon is the limit of the present model for $\Delta\omega \to 0$ and, possibly, $r_0 \to 0$.

6.7 Properties as a wave packet

Clearly, the solution we have explained forms a true wave packet. In the sense of communications theory [28, 29], the complex amplitude contains the information while Ω is simply the frequency of the carrier. Interpreted as a function of t, a signal such as (6.54) is simultaneously amplitude modulated with envelope $e^{-A(\tau)}$ and frequency modulated with instantaneous frequency

$$\dot{\varphi}(\tau) = \frac{\partial\varphi(\tau)}{\partial\tau} = \Omega - \dot{B}(\tau) = \Omega - \tau k \sin b(\tau),$$

where b is as occurring in (E.21). The dot on top of a letter denotes $\partial/\partial t$, as usual, but due to (6.4) this is equivalent to $\partial/\partial\tau$.

Consider next a dispersive medium presenting a linear dispersion in the relevant range $\Omega - \Delta\omega/2 < \omega < \Omega + \Delta\omega/2$. We assume the medium to be uniform and the signal to be travelling in the x-direction so that the y- and z-directions are irrelevant. We may then immediately draw

from the results presented in Appendix E3. We make use of the notation defined there, i.e., we denote by $f(t)$ the signal at the beginning of the medium, say at $x=0$, and by $g(t,x)$ the resulting signal at position $x \geq 0$, with thus $f(t)=g(t,0)$.

If the signal we are analysing were indeed a photon, it would be either fully absorbed in the medium or fully transmitted. Since only the latter alternative is of interest to us, we may set $a_0=0$, in particular in (E.32)(a). If (E.32)(b) is also fulfilled, at least with sufficient accuracy, we may apply in particular (E.33) and (E.34), with $a_0=0$, ω_0 replaced by Ω, and g standing for any of the presently relevant field quantities. It follows then clearly from (E.34) that the "carrier signal" $e^{j\Omega t}$ propagates with the phase velocity $v_{ph}(\Omega)$ (cf. (E.35)), while the propagation of the complex amplitudes does take place with the group velocity v_g. But according to (6.48) the total energy depends solely on the moduli of such complex amplitudes. Hence, as follows from (E.36)(b) and $a_0=0$, the energy does indeed travel with the group velocity v_g. As recalled in Appendix E, however, the definition $v_g=\mathrm{d}\omega/\mathrm{d}k$ (cf. (E.37)), where k is the wave number, is an appropriate choice only under restrictive circumstances that are commonly ignored.

A last comment concerns the cancellation of the oscillatory terms in (6.47). As discussed for arriving at (6.48), that cancellation is not perfect and $\widetilde{\hbar}$ is therefore not strictly constant. This puts a limit on the accuracy by which $\widetilde{\hbar}$ can be specified, and in that sense, a limit on the validity of the Planck and Einstein laws, the error being apparently of the order of $(\Delta\omega/\Omega)^2$.

6.8 Further problems

As we have seen, the localized fields discussed in Sections 5 and 6 are described by sets of equations that are, at least originally, underdetermined. Some of the resulting indeterminacy like the one for χ_y and χ_z (Subsection 6.5) is irrelevant, at least in as far as energy is concerned.

It could well be, however, that any freedom that may finally remain must be used for ensuring stability of the field configuration (assuming that stable configurations indeed exist). Approaching this important aspect requires examining not simply the configuration at equilibrium, i.e., the steady-state behaviour, as we have done, but the full underlying dynamical problem. This is a truly difficult and mathematically challenging issue that must be left for future studies. Helpful for addressing it could be the concept of multidimensionally passive systems. The theory of such systems has been developed in the context of finding a robust method for numerically integrating PDEs that describe physically passive dynamical systems [18]. This method is based on a physically motivated approach for constructing mathematical algorithms that are suitable for robust one-dimensional and multidimensional digital filtering [16]. Of particular relevance for our purpose could be the concept of local internal multidimensional passivity that is briefly addressed in [18].

Alternatively, or complementary to the interpretation just given, it could also be that there is indeed some true indeterminism in the precise configuration of the fields we have examined. This would, for instance, appear to facilitate the known interaction between particles and thus the reconfiguration of the fields after such an interaction has taken place.

6.9 Relation between electron, positron, and photon

A reasonable relation between an electron, a positron, and a photon appears possible only if the latter is circularly polarized. This will therefore be assumed hereafter. We also assume the fields analysed in Sections 5 and 6 to be indeed valid models of an electron/positron and a photon, respectively, in which case we may set in particular, $\tilde{\hbar} = \hbar$.

It is instructive to observe both types of fields in their respective basic reference frame, RF_B. For an electron or positron, RF_B is the reference frame we have adopted in Section 5, i.e., the one in which the field, although internally rotating, is at rest as a whole, thus if simply observed from the outside. A photon, on the contrary, always moves with the speed of light and thus can never be at rest in the sense just defined. It is appropriate therefore to define the basic reference frame RF_B of a photon as being that RF in which the total energy, W_B, is equal to the threshold beyond which the photon can produce an electron-positron

pair, thus for which $\Omega = \Omega_B$, with $\Omega_B = W_B / 2\hbar$. Obviously, a photon may be assumed to be unique in the same sense as an electron and a positron provided all particles are observed in their respective basic reference frame.

Consider thus a photon in its specific RF_B. We may assume it to consist of an electron and a positron whose charges are interleaved. The total energy is equal to the combined original energies. We may also assume that the positive and negative charges are rotating in the same direction, which amounts to a doubling of the angular momentum, thus of the spin. Although the existence of such an angular momentum requires the internal presence of non-vanishing lateral (i.e., with respect to the direction of propagation) velocity components, these are, as explained in Subsection 3.8, undetectable for an external observer for whom the photon is moving at the speed of light (cf. (3.51)), . On the other hand, due to the opposite signs of the electron and positron charges, the resulting currents are oppositely equal. Thus if observed from a distance, the total magnetic moment appears to sum up to zero in the same way as the total electric charge.

Finally, let us still write the relation (5.32), which we had obtained for the kinetic energy of an electron, more specifically in the form $W_k = \bar{\omega} L_{el}$, where L_{el} is the spin of the electron and $\bar{\omega}$ its nominal angular velocity. Assume that (5.32) expresses in fact a more universal law. Since for a photon the rest energy $W_0 = 0$, its kinetic energy coincides with its total energy W and we therefore could write $W = W_k = \Omega L_{ph}$, where L_{ph} is the photon spin. Due to (6.49)(a) we could thus conclude that $L_{ph} = \hbar$, as is indeed known to be the case. On the other hand, the doubling of the spin mentioned in the previous paragraph allows us to state $L_{ph} = 2L_{el}$, which leads immediately to $L_{el} = \hbar / 2$. Such results are compatible with the assumption that the energy of the photon, thus its kinetic energy, does in fact consist completely of its inner rotational energy. In other words, the entire photon energy would be due to a movement that is unobservable from the outside.

7. Moving particles

7.1 Electron-like particle observed in different reference frames

7.1.1 A first general relation

Let there be given, in a reference frame RF, a field that is described by equations precisely as discussed in Section 5. We consider such a field to be an entity of its own, refer to it by calling it a particle, say Pa, and say that Pa is at rest in RF. There is of course internal rotational movement, but at any fixed position x, y, z the field is independent of t.

Let us first determine the total momentum of Pa (cf. (4.7)),

$$\mathbf{J} = (J_x, J_y, J_z)^{\mathrm{T}} = \int_V \mathbf{j}\,\mathrm{d}V, \quad \mathbf{j} = (j_x, j_y, j_z)^{\mathrm{T}} = \frac{2w_0}{c\alpha^2}\boldsymbol{\beta}.$$

For this it is convenient to express the Cartesian components of $\mathbf{j}$ in terms of those in spherical coordinates. Since $j_r = j_\theta = 0$ and $j_\varphi = j$ (cf. (5.21)) we derive from (A.13),

$$j_x = 0, \quad j_y = -j\sin\varphi, \quad j_z = j\cos\varphi,$$

and since j is independent of φ we obtain

$$\mathbf{J} = \int_V (0, -\sin\varphi, \cos\varphi)^{\mathrm{T}} jr^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi = \mathbf{0}. \tag{7.1}$$

Hence, $\mathbf{J}$ is strictly zero.

Next, let us observe Pa in a further reference frame RF' such as the one in Sections 2 and 3, thus with RF' moving with constant velocity $\mathbf{v}_0 = (v_0, 0, 0)^{\mathrm{T}}$ (cf. (2.3)) with respect to RF. We may then make use of (3.8), or better directly of the previous equation (3.6), from which (3.8) is derived. Taking into account (4.3) and the corresponding relation for S'_x we may express (3.6) in the form

$$w' + c\boldsymbol{\beta}_0^{\mathrm{T}}\mathbf{j}' = w - c\boldsymbol{\beta}_0^{\mathrm{T}}\mathbf{j}, \tag{7.2}$$

which we have written in such a way that it is valid for any $\mathbf{v}_0$. Choosing then an arbitrary fixed value of t', integrating (7.2) over the entire relevant volume V' in RF', and taking into account (3.7), we obtain

$$W' + c\boldsymbol{\beta}_0^{\mathrm{T}}\mathbf{J}' = \int_{V'} w\,\mathrm{d}V' - c\boldsymbol{\beta}_0^{\mathrm{T}}\int_{V'} \mathbf{j}\,\mathrm{d}V', \tag{7.3}$$

where

$$\left.\begin{aligned} W' &= \int_{V'} w' \mathrm{d}V', \quad \mathrm{d}V' = \mathrm{d}x'\mathrm{d}y'\mathrm{d}z', \quad \text{(a)} \\ \mathbf{J}' &= \int_{V'} \mathbf{j}' \mathrm{d}V' = \frac{2}{c}\int_{V'} \frac{w_0}{\alpha'^2}\boldsymbol{\beta}' \mathrm{d}V'. \qquad \text{(b)} \end{aligned}\right\} \tag{7.4}$$

Let us then replace the two integrations that appear explicitly in (7.3) by integrations over the corresponding volume V in RF. In view of the last equality in (2.4)(a), we have indeed

$$t = \alpha_0 t' + \beta_0 \frac{x}{c}, \tag{7.5}$$

where t' is fixed. As is known, we then have

$$\mathrm{d}V' = \alpha_0 \mathrm{d}V, \quad \mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Hence, considering w and $\mathbf{j}$ to be functions of x, y, z, and t, we may write

$$\left.\begin{aligned} \int_{V'} w \,\mathrm{d}V' &= \alpha_0 \int_V w(x,y,z,t)\,\mathrm{d}V, \\ \int_{V'} \mathbf{j}\,\mathrm{d}V' &= \alpha_0 \int_V \mathbf{j}(x,y,z,t)\,\mathrm{d}V, \end{aligned}\right\} \tag{7.6}$$

where t has to be chosen according to (7.5), i.e., with t' constant and x variable.

Since we assume the particle to be at rest in RF, thus to be there as mentioned above, we obtain from (7.6),

$$\left.\begin{aligned} \int_{V'} w \,\mathrm{d}V' &= \alpha_0 W, \quad W = \int_V w(x,y,z,t)\,\mathrm{d}V, \quad \text{(a),(b)} \\ \int_{V'} \mathbf{j}\,\mathrm{d}V' &= \alpha_0 \mathbf{J}, \quad \mathbf{J} = \int_V \mathbf{j}(x,y,z,t)\,\mathrm{d}V, \qquad \text{(c),(d)} \end{aligned}\right\} \tag{7.7}$$

where $w(x,y,z,t)$ and $\mathbf{j}(x,y,z,t)$ are in fact independent of t. Consequently, W and the components of $\mathbf{J}$ are true constants. In view of (7.1), (7.3) yields

$$W' + c\boldsymbol{\beta}_0^{\mathrm{T}}\mathbf{J}' = \alpha_0 W. \tag{7.8}$$

For an observer in RF′ the particle Pa is moving with velocity $-\mathbf{v}_0$ with respect to RF′. From the point of view of this observer it is therefore appropriate to rewrite (7.8) in the form

$$W_p - c\boldsymbol{\beta}_p^{\mathrm{T}}\mathbf{J}_p = \alpha_p W_{p0}, \tag{7.9}$$

where $W_p = W'$ is the total energy of the particle, $W_{p0} = W$ its rest energy, $\mathbf{J}_p = \mathbf{J}'$ its momentum, and $\mathbf{v}_p = -\mathbf{v}_0$ its velocity. Let M_{p0} be its rest mass. According to classical relativistic dynamics we should have

$$\left.\begin{aligned} &W_p = \frac{1}{\alpha_p} W_{p0}, \quad \mathbf{J}_p = \frac{\mathbf{v}_p}{c^2} W_p, \quad W_{p0} = c^2 M_{p0}, \quad \text{(a),(b),(c)} \\ &\alpha_p = \sqrt{1-\beta_p^2}, \quad \beta_p^2 = \boldsymbol{\beta}_p^{\mathrm{T}} \boldsymbol{\beta}_p, \quad \boldsymbol{\beta}_p = \frac{\mathbf{v}_p}{c}. \end{aligned}\right\} \tag{7.10}$$

As can easily be verified, (7.10) is indeed a solution of (7.9). On the contrary, (7.9) can be shown to be violated if instead of (7.10) we would assume the alternative relativistic dynamics to hold.

7.1.2 A second general relation

We start from (3.13)(a), which has indeed been written in such a way that it holds for any $\mathbf{v}_0$. Using

$$\frac{2}{\alpha^2} = 1 + \frac{1+\beta^2}{\alpha^2}, \quad \frac{2}{\alpha'^2} = 1 + \frac{1+\beta'^2}{\alpha'^2},$$

it can be expressed in the form,

$$\alpha_0^2 \frac{1+\beta'^2}{\alpha'^2} = \beta_0^2 + \frac{1+\beta^2}{\alpha^2} - 4\frac{\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\beta}}{\alpha^2} + 2\frac{(\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\beta})^2}{\alpha^2} - 2(\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\gamma})^2,$$

which, after multiplying it by w_0 and taking into account (2.26) and (3.5), gives rise to

$$\alpha_0^2 w' = w + \beta_0^2 w_0 - 4w_0 \frac{\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\beta}}{\alpha^2} + 2w_0 \frac{(\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\beta})^2}{\alpha^2} - 2w_0(\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\gamma})^2.$$

Integrating this over the entire volume V' and proceeding as done for passing from (7.2) to (7.8) we obtain after taking into account (7.1) and dividing by α_0,

$$\alpha_0 W' = W + \beta_0^2 W_0 + 2\int_V w_0 \frac{(\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\beta})^2}{\alpha^2}\,\mathrm{d}V - 2\int_V w_0(\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\gamma})^2\,\mathrm{d}V, \tag{7.11}$$

where W' and W are as defined by (7.4)(a) and (7.7)(b), and where W_0 is the total energy of the rest field in RF , i.e.,

$$W_0 = \int_V w_0 \mathrm{d}V. \tag{7.12}$$

The vectors $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ can each be decomposed into the sum of two respective parts, a longitudinal vector that is co-parallel with $\boldsymbol{\beta}_0$ and a lateral vector that is orthogonal to $\boldsymbol{\beta}_0$. We denote the former by $\boldsymbol{\beta}_\parallel$ and $\boldsymbol{\gamma}_\parallel$, and the latter by $\boldsymbol{\beta}_\perp$ and $\boldsymbol{\gamma}_\perp$. Let $\beta_\parallel$, $\gamma_\parallel$, $\beta_\perp$, and $\gamma_\perp$ be the lengths of those partial vectors. We have,

$$\left.\begin{aligned} (\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\beta})^2 = \beta_0^2\beta_\parallel^2, \quad (\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\gamma})^2 = \beta_0^2\gamma_\parallel^2, \\ \beta_\parallel^2 + \beta_\perp^2 = \beta^2, \quad \gamma_\parallel^2 + \gamma_\perp^2 = 1. \end{aligned}\right\} \tag{7.13}$$

Hence, we derive from (7.11),

$$\left.\begin{aligned} \alpha_0 W' &= W + \beta_0^2(W_{0\perp} - W_{0\parallel} + W_{k\parallel}) \quad \text{(a)} \\ &= W + \beta_0^2(2W_{0\perp} - W_0 + W_{k\parallel}), \quad \text{(b)} \end{aligned}\right\} \tag{7.14}$$

where (cf. (7.12))

$$W_{0\parallel} = \int_V w_0\gamma_\parallel^2 \mathrm{d}V = W_0 - W_{0\perp}, \quad W_{0\perp} = \int_V w_0\gamma_\perp^2 \mathrm{d}V, \tag{7.15}$$

$$W_{k\parallel} = 2\int_V w_0 \frac{\beta_\parallel^2}{\alpha^2}\mathrm{d}V. \tag{7.16}$$

Clearly, $W_{0\parallel}$ and $W_{0\perp}$ may be interpreted as the rest-field energies that are due, respectively, to the *longitudinal* and the *transversal* components of the electric and the magnetic rest fields in RF, while $W_{k\parallel}$ is the *longitudinal* part of the kinetic energy of the actual field in RF (cf. (4.34)).

Let us now assume more specifically that the axis of rotation of the field is parallel to $\mathbf{v}_0$, say coincident with the $x-$axis. The equations derived in Section 5 then apply directly to the present field in RF. In accordance with (5.1), which is written in terms of spherical coordinates, we therefore have $\beta_\parallel = 0$ everywhere and thus $W_{k\parallel} = 0$ (cf. (7.16)). If we then adopt again the same notation as we have done above subsequently to (7.8), we derive from (7.14) and (7.15),

$$\alpha_p W_p - W_{p0} = \beta_p^2(W_{0ra} - W_{0ax}), \tag{7.17}$$

where W_{0ra} and W_{0ax} are rest field energies that are inherent to the particle Pa itself. More precisely, W_{0ax} is the *axial* part of W_0, i.e. that part of W_0 that is due to the components of $\mathbf{E}_0$ and $\mathbf{H}_0$ parallel to the axis of

rotation of Pa. Furthermore, since $\gamma_\varphi = 0$ (cf. (5.2)), $W_{0ra} = W_0 - W_{0ax}$ is due to the components of $\mathbf{E}_0$ and $\mathbf{H}_0$ perpendicular to that axis and may therefore be called the *radial* part of W_0. If

$$W_{0ra} = W_{0ax} = \frac{1}{2} W_0 , \tag{7.18}$$

(7.17) would agree, and this irrespective of the value of β_p, with (7.10)(a) and thus, in view of (7.9), also with (7.10)(b). If (7.18) must indeed hold, it might have to be added to the requirements listed in Appendix D.

7.1.3 Some further details

We want to evaluate again the expressions appearing between parentheses in (7.14), but this time without requiring the axis of rotation of the field to be parallel to $\mathbf{v}_0$. For doing this we adopt for RF simultaneously a spherical and a Cartesian coordinate system. We assume the former to be just like the one in Section 5. As for the latter, we choose it in such a way that the positive $x-$axis coincides with the positive axis of rotation ($\theta = 0$), that the plane $z = 0$ coincides with the one determined by the $x-$axis and the vector $\boldsymbol{\beta}_0 = (\beta_{ox}, \beta_{0y}, \beta_{0z})^{\mathrm{T}}$, in which case we have $\beta_{oz} = 0$, and that $\beta_{0y} > 0$, due to which we have $0 < \theta_0 < \pi$ for the angle θ_0 between the positive $x-$axis and the positive direction of $\boldsymbol{\beta}_0$. In particular we may write,

$$\beta_{0x} = \beta_0 \cos\theta_0 , \quad \beta_{0y} = \beta_0 \sin\theta_0 , \quad \beta_{0z} = 0 . \tag{7.19}$$

On the other hand, since $\beta_r = \beta_\theta = \gamma_\varphi = 0$ and $\beta_\varphi = \beta$, we obtain from (A.13),

$$\beta_x = 0, \quad \beta_y = -\beta \sin\varphi, \quad \beta_z = \beta \cos\varphi , \tag{7.20}$$

$$\gamma_x = \gamma_{ax} , \quad \gamma_y = \gamma_{ra} \cos\varphi, \quad \gamma_z = \gamma_{ra} \sin\varphi , \tag{7.21}$$

where

$$\left.\begin{aligned} &\gamma_{ax} = \gamma_r \cos\theta - \gamma_\theta \sin\theta, \quad \gamma_{ra} = \gamma_r \sin\theta + \gamma_\theta \cos\theta, \\ &\gamma_{ax}^2 + \gamma_{ra}^2 = \gamma_r^2 + \gamma_\theta^2 = 1. \end{aligned}\right\} \tag{7.22}$$

As can be verified, γ_{ax} is indeed the axial component of $\boldsymbol{\gamma}$, and γ_{ra} its radial component. From (7.19) to (7.21) we deduce

$$\left.\begin{aligned}\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\beta} &= -\beta_0\beta\sin\theta_0\sin\varphi, && \text{(a)}\\ \boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\gamma} &= \beta_0(\gamma_{ax}\cos\theta_0+\gamma_{ra}\cos\varphi\sin\theta_0). && \text{(b)}\end{aligned}\right\} \tag{7.23}$$

These expressions remain valid in the limits $\theta_0 = 0$ and $\theta_0 = \pi$.

We now proceed to achieving our present goal. Instead of starting from (7.14), to which we have referred above, it is more convenient to make use of the original expression (7.11). The integrals in this depend on φ only via the factors $\sin\varphi$ and $\cos\varphi$ appearing in (7.23). Hence, replacing $\sin^2\varphi$ by $(1-\cos 2\varphi)/2$ and $\cos^2\varphi$ by $(1+\cos 2\varphi)/2$ one derives from (7.23),

$$2\int_V w_0\frac{(\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\beta})^2}{\alpha^2}\mathrm{d}V = \frac{1}{2}\beta_0^2 W_k\sin^2\theta_0,$$

$$2\int_V w_0(\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\gamma})^2\mathrm{d}V = \beta_0^2(2W_{0ax}\cos^2\theta_0 + W_{0ra}\sin^2\theta_0),$$

where (cf. (4.34)(b))

$$\left.\begin{aligned} W_k &= W - W_0 = 2\int_V w_0\frac{\beta^2}{\alpha^2}\mathrm{d}V, \quad W_{0ax} = \int_V w_0\gamma_{ax}^2\mathrm{d}V, && \text{(a)}\\ W_{0ra} &= W_0 - W_{0ax} = \int_V w_0\gamma_{ra}^2\mathrm{d}V, && \text{(b)}\end{aligned}\right\} \tag{7.24}$$

and where (7.24)(b) can be verified by means of (7.22). Consequently, (7.11) and thus (7.14) may be written in the form

$$\begin{aligned}\alpha_0 W' &= W + \beta_0^2\left(W_0 + \frac{1}{2}W_k\sin^2\theta_0 - 2W_{0ax}\cos^2\theta_0 - W_{0ra}\sin^2\theta_0\right)\\ &= W + \frac{1}{2}\beta_0^2[W\sin^2\theta_0 + (W_{0ra}-W_{0ax})(2\cos^2\theta_0 - \sin^2\theta_0)].\end{aligned} \tag{7.25}$$

More specifically, if (7.18) holds, (7.25) simplifies to

$$\alpha_0 W' = W + \frac{1}{2}\beta_0^2 W\sin^2\theta_0. \tag{7.26}$$

According to (7.26), thus if (7.18) holds, (7.17) generalizes to

$$\frac{\alpha_p W_p - W_{p0}}{W_{p0}} = \frac{1}{2}\beta_p^2\sin^2\theta_0. \tag{7.27}$$

In (7.27), the left-hand side represents the relative error by which (7.10)(a) is satisfied. In view of the right-hand side of (7.27), that error is

extremely small if $|\beta_p|$ and $|\sin\theta_0|$ are both small. However, if all terms in (7.25) must be retained, (7.27) has to be replaced by

$$\frac{\alpha_p W_p - W_{p0}}{W_{p0}} = \frac{1}{2}\beta_p^2[\sin^2\theta_0 + \frac{(W_{0ra} - W_{0ax})}{W_{p0}}(2\cos^2\theta_0 - \sin^2\theta_0)]. \quad (7.28)$$

If $\sin^2\theta_0$ increases from 0 to 1, the expression between brackets in (7.28) varies monotonically from 2ρ to $1-\rho$ where $\rho := (W_{0ra} - W_{0ax})/W_{p0}$. It is thus always non-negative if $0 \le \rho \le 1$. Since W_{p0} corresponds to W in Subsection 5.3 and is thus given by (5.19)(a), where $w = w_0/\alpha$, we have, $W_{p0} > W_0 = W_{0ra} + W_{0ax}$ and therefore definitely $|\rho| < 1$, but $\rho \ge 0$ requires $W_{0ra} \ge W_{0ax}$. In any case, the modulus of the entire right-hand side of (7.28) is always < 1.

7.1.4 Charge and current densities

Referring to the discussion just above we first assume $\theta_0 = 0$, i.e., we assume the axis of rotation and the direction of $\mathbf{v}_0$ to be both parallel to the $x-$axis. According to (A.13) and to what we have seen in Section 5, we may then write for the current density in RF, $\mathbf{i} = (i_x, i_y, i_z)^{\mathrm{T}}$,

$$i_x = 0, \quad i_y = -i\sin\varphi, \quad i_z = i\cos\varphi,$$

where i is independent of φ. Hence,

$$\int_V \mathbf{i}\,\mathrm{d}V = \mathbf{0}. \quad (7.29)$$

This relation remains true of course for any orientation of the axis of rotation.

Independently of that orientation, we may continue choosing $\mathbf{v}_0$ according to (2.3). For the current density in RF′, $\mathbf{i}' = (i_x', i_y', i_z')^{\mathrm{T}}$, we then find from (2.7) and (7.29), making again use of $\mathrm{d}V' = \alpha_0 \mathrm{d}V$,

$$\left.\int_{V'} \mathbf{i}'\mathrm{d}V' = \alpha_0 \int_V \mathbf{i}'\mathrm{d}V = -\mathbf{v}_0 Q, \quad Q = \int_V q\,\mathrm{d}V. \quad \text{(a), (b)}\right\} \quad (7.30)$$

In view of the definition $\mathbf{v}_p = -\mathbf{v}_0$ adopted in the previous subsections, the result of (7.30)(a) can be written in the form

$$\int_{V'} \mathbf{i}' \mathrm{d}V' = \mathbf{v}_p Q .$$

The quantity Q defined by (7.30)(b), clearly, is the total charge. As is known (and as is easily verified by means of (2.7)(d)) it is independent of the selected reference frame.

7.2 Behaviour in the presence of an external electrostatic field

7.2.1 Dynamic equations of an electron-like model

We consider again an EM field, say EF, that occupies a relevant volume V in a reference frame RF. We want to examine the influence a superposed external EM field EFe, thus a field characterized in principle by, say, $\mathbf{E}_e$, $\mathbf{H}_e$, q_e, and $\mathbf{i}_e$, has upon EF. We do restrict ourselves to a purely electrostatic EFe and assume furthermore that the charges associated with EFe are sufficiently far removed from the region of interest so that they vanish everywhere inside of V. For all points of relevance we may therefore write,

$$\frac{\partial \mathbf{E}_e}{\partial t} = \mathbf{0}, \quad \nabla \times \mathbf{E}_e = \mathbf{0}, \quad \nabla^{\mathrm{T}} \mathbf{E}_e = q_e = 0, \quad \mathbf{H}_e = \mathbf{i}_e = \mathbf{0} . \tag{7.31}$$

If we then replace $\mathbf{E}$, $\mathbf{H}$, q, and $\mathbf{i}$ in (2.1) and (2.2) by an additive combination of EF and EFe, and take into account (7.31), we find that inside of V the original equations (2.1) and (2.2) are simply restored. This shows that the configurations the field EF, thus the field characterized by $\mathbf{E}$, $\mathbf{H}$, q, and $\mathbf{i}$, can admit are indeed the same as in the absence of EFe.

We now assume more specifically that EF constitutes in fact a particle, Pa, obviously of small size. We may therefore further idealize the external field EFe and require $\mathbf{E}_e$ to be uniform, thus to be independent not only of t but also of $\mathbf{r} = (x, y, z)^{\mathrm{T}}$. We may then assume the coordinate axes to be chosen in such a way that

$$\mathbf{E}_e = (E_{ex}, 0, 0)^{\mathrm{T}} \quad \forall \mathbf{r}, t , \tag{7.32}$$

with E_{ex} independent of t and $\mathbf{r}$. As in Subsection 7.1, we are interested more precisely in a particle Pa originating from a field EF such as the one analysed in Section 5. We adopt again the terminology defined in Subsection 7.1. We therefore say again that Pa is at rest in a given reference frame if it is there exactly as described in Section 5 (which obviously still implies the presence of internal rotational flows of the EM

fluid), thus as assumed to be the case in the reference frame RF considered in Subsection 7.1.1. We restrict the analysis to the case where the axis of rotation of Pa is coincident with the $x-$axis of RF and thus parallel to $\mathbf{E}_e$ (cf. the discussion in Subsection 7.1.2).

Let Pa indeed be at rest in RF at some initial time t_0. For $t > t_0$ we assume it be subject to EFe and, as a consequence, be travelling as a whole, thus as an entity of its own, with velocity $\mathbf{v}_p = \mathbf{v}_p(t)$ in the direction of $\mathbf{E}_e$, thus with

$$\mathbf{v}_p = (v_p, 0, 0)^{\mathrm{T}}. \tag{7.33}$$

Since, as we have seen, the equations describing EF are independent of EFe we may assume that for any $t > t_0$ the field EF admits in RF that configuration that one obtains after Lorentz transforming the field from a reference frame RF' where Pa is at rest. The constant velocity by which RF' is moving with respect to RF must then be chosen according to $\mathbf{v}_0 = \mathbf{v}_p(t)$ where t is indeed the specific time instant under consideration in RF. We may therefore refer to Subsection 7.1 for determining the relevant quantities characterizing Pa. We must be careful however, when doing this because the roles of RF and RF' are now exactly interchanged. Our present notation has been adopted in order not to burden it by having to add primes to all expressions of major interest; the same kind of consideration had led to the opposite choice in Subsection 7.1.

Since in RF, EF satisfies (2.1) and (2.2), it also satisfies there equations such as (4.14)(a) where all quantities are as defined in Subsection 4.1. We may however not simply interpret the derivative $\partial\mathbf{j}/\partial t$, thus the first term in (4.14)(a), as being the actual rate of change of the momentum density $\mathbf{j}$. In fact, (4.14)(a) only states how large that rate would be if EFe were absent while the other three terms were as is actually the case; the first one of these three terms is due to the convection of the EM fluid that corresponds to EF, while the remaining two represent the effect of the internal forces acting in EF. In reality, the full rate $\partial\mathbf{j}/\partial t$ is also influenced by the external force EFe exerts upon EF. The density of that external force is known to be $\mathbf{f}_e = q\mathbf{E}_e$. Hence, in order to obtain the full rate we must replace (4.14)(a) by

$$\frac{\partial \mathbf{j}}{\partial t} + \left(\nabla^{\mathrm{T}}(\mathbf{v}\mathbf{j}^{\mathrm{T}})\right)^{\mathrm{T}} + (\nabla^{\mathrm{T}}\mathbf{T}_0)^{\mathrm{T}} + \mathbf{f}_c = q\mathbf{E}_e\,. \tag{7.34}$$

Clearly, the volume force density $q\mathbf{E}_e$ is analogous to $q\mathbf{E}$ in (4.4).

Let us then integrate (7.34) over a sufficiently and adequately large volume V. For calculating the result we may make use of (4.8) and thus replace the contribution of the second and third terms in the left-hand side of (7.34) by

$$\int_V (\nabla^{\mathrm{T}}\mathbf{T}_c)^{\mathrm{T}}\,\mathrm{d}V \tag{7.35}$$

where $\mathbf{T}_c$ is given by (2.9) and (4.5). Since $\mathbf{E}$ decreases as fast as the square of the distance from, roughly, the "centre" of EF, and since $\mathbf{H}$ decreases even faster, application of Gauss' theorem shows that the total contribution of the second and third terms in the left-hand side of (7.34) is zero. As will be proved in Subsection 7.2.2, the last term in the left-hand side of (7.34) does not contribute either, and this as a consequence of the circular and equatorial symmetries Pa has in RF′.

In RF, let then $\mathbf{J}_p$ be the total momentum of Pa, W_p its total energy, and W_{p0} its rest energy, i.e.,

$$\mathbf{J}_p = \int_V \mathbf{j}\,\mathrm{d}V, \quad W_p = \int_V w\,\mathrm{d}V, \quad W_{p0} = W_p\,|_{\mathbf{v}_p=\mathbf{0}}\,.$$

Since $\mathbf{E}_e$ is assumed independent of $\mathbf{r}$, the total force $\mathbf{F}_e$ that EMFe exerts upon Pa is given by

$$\mathbf{F}_e = \int_V q\mathbf{E}_e\,\mathrm{d}V = Q\mathbf{E}_e\,, \tag{7.36}$$

where Q is the total charge, as before. Altogether, finally, the simple relation

$$\frac{\mathrm{d}\mathbf{J}_p}{\mathrm{d}t} = \mathbf{F}_e \tag{7.37}$$

is deduced by integrating (7.34) as stated, $\mathbf{F}_e$ being as given by (7.36).

Consider now (4.14)(b). We can make about the role of $\partial w/\partial t$ exactly the same kind of general remarks as previously about that of $\partial\mathbf{j}/\partial t$. Hence, we must now complement the term $\mathbf{i}^{\mathrm{T}}\mathbf{E}$ by a corresponding term that takes into account the work done by EFe and therefore replace (4.14)(b) by

$$\frac{\partial w}{\partial t} + \nabla^{\mathrm{T}}(w\mathbf{v}) + \nabla^{\mathrm{T}}(\mathbf{T}_0\mathbf{v}) + \mathbf{i}^{\mathrm{T}}\mathbf{E} = \mathbf{i}^{\mathrm{T}}\mathbf{E}_e, \tag{7.38}$$

where in fact we have $\mathbf{i}^{\mathrm{T}}\mathbf{E}_e = i_x E_{ex}$. As will be seen in Subsection 7.2.2, the $x-$component of $\mathbf{i}$ is actually given by $i_x = qv_p$, where v_p is as in (7.33). Although $\mathbf{i} \neq q\mathbf{v}$ (i_y and i_z are indeed not vanishing), we may, due to (7.32), nevertheless write $i_x E_{ex} = q\mathbf{v}_p^{\mathrm{T}}\mathbf{E}_e$ and thus replace (7.38) by

$$\frac{\partial w}{\partial t} + \nabla^{\mathrm{T}}(w\mathbf{v}) + \nabla^{\mathrm{T}}(\mathbf{T}_0\mathbf{v}) + \mathbf{i}^{\mathrm{T}}\mathbf{E} = q\mathbf{v}_p^{\mathrm{T}}\mathbf{E}_e. \tag{7.39}$$

Next, we integrate (7.39) over the volume V, as we have done for (7.34). According to (4.12), the two inner terms in the left-hand side of (7.39) may be replaced by $\nabla^{\mathrm{T}}\mathbf{S} = \nabla^{\mathrm{T}}(\mathbf{E}\times\mathbf{H})$ and thus yield zero, and this for the same reason as for (7.35). As will be seen in Subsection 7.2.2, the last term in the left-hand side of (7.39) also yields zero. We thus obtain altogether

$$\frac{\mathrm{d}W_p}{\mathrm{d}t} = \mathbf{v}_p^{\mathrm{T}}\mathbf{F}_e, \tag{7.40}$$

where $\mathbf{F}_e$ is the total external force acting on Pa, as defined by (7.36).

As already pointed out by Einstein [1] in the classical context, the movement of a particle under the influence of some other field (for instance gravitation) must follow the same general rules as in the case of an electric field. Hence, (7.37) and (7.40) must hold also if $\mathbf{F}_e$ is not simply given by (7.36) but is a force of more general nature. This explains why all known experiments with EM particles are in agreement with the predictions of classical relativistic dynamics, contrary to what one might expect after taking a first look at the role of the alternative theory at the basic level. There remains of course an open question concerning the inclination θ_0 introduced in Subsection 7.1.3. Does this have an effect and, if so, can such an effect be observed experimentally?

7.2.2 Outline of supplementary proofs

We assume RF and RF′ to be as defined in Subsection 7.2.1, thus with the roles of RF and RF′ interchanged compared to the notation adopted in Subsection 7.1. Hence, for the same reasons as explained there, we may now replace integrations in RF by integrations in RF′, but with

$\mathrm{d}V = \alpha_0 \mathrm{d}V'$. We will make intensive use of Appendix D2, but, without further mentioning it, we assume systematically that primes have been added as needed to the equations listed there. We also refer to the following equations obtained by inverting (2.5) and (2.6), i.e.,

$$\left.\begin{aligned} E_y &= \frac{1}{\alpha_0}(E'_y + \mu v_0 H'_z), \quad E_z = \frac{1}{\alpha_0}(E'_z - \mu v_0 H'_y), \quad \text{(a),(b)} \\ H_y &= \frac{1}{\alpha_0}(\mathrm{H}'_y - \varepsilon v_0 E'_z), \quad H_z = \frac{1}{\alpha_0}(H'_z + \varepsilon v_0 E'_y). \quad \text{(c),(d)} \end{aligned}\right\} \tag{7.41}$$

Consider first the term $q\mathbf{E}$ that occurs in $\mathbf{f}_c$. Since Pa is at rest in RF', we have there, $i'_x = 0$, and we therefore deduce from (2.7) the first four of the following equalities,

$$i_x = q v_0, \quad i_y = i'_y, \quad i_z = i'_z, \quad q = \frac{q'}{\alpha_0}, \quad E_x = E'_x, \tag{7.42}$$

the last one holding in view of (2.5)(a). Hence, we obtain for the $x-$component,

$$\int_V q E_x \mathrm{d}V = \int_{V'} q' E'_x \mathrm{d}V'. \tag{7.43}$$

Due to the symmetry the field admits in RF' that last integral can be seen to vanish (cf. (D.13)(a)). For the $y-$component we find, making use of (7.41)(a),

$$\alpha_0 \int_V q E_y \mathrm{d}V = \int_{V'} q' E'_y \mathrm{d}V' + \mu v_0 \int_{V'} q' H'_z \mathrm{d}V',$$

where both integrals to the right vanish for similar symmetry reasons, and the same result holds correspondingly for the $z-$component (cf. (D.13)(b)) and (c) as well as (D.14)(b) and (c).

For the $x-$component of the second term in $\mathbf{f}_c$ we are led, using (2.7)(b) and (c) as well as (7.41)(c) and (d), to the integration

$$\begin{aligned} \alpha_0 \int_V (i_y H_z - i_z H_y)\mathrm{d}V = &\int_{V'} (i'_y H'_y - i'_z H'_z)\mathrm{d}V' \\ &- \varepsilon v_0 \int_{V'} (i'_y E'_z + i'_z E'_y)\mathrm{d}V'. \end{aligned} \tag{7.44}$$

Consider first the term $i'_y H'_y$. According to (D.12)(b) and (D.14)(b), it is positive or negative depending on the sign of $\varepsilon_x \varepsilon_y \varepsilon_z$, thus equally often for the same value of $|i'_y H'_y|$, and the same holds for $i'_z H'_z$. Hence, the

first integral in the right-hand side of ((7.44) vanishes. As for $i'_y E'_z + i'_z E'_y$, we conclude from (D.15)(b) and (D.17)(a) that for each pair of positions (cf. the sentence following (D.17) in Appendix D) the value of $i'_y E'_z$ is equal to minus the value of $i'_z E'_y$ at the complementary position. Hence, the second integral in the right-hand side of (7.44) also vanishes.

We can now be brief about the term $\mathbf{i}^{\mathrm{T}}\mathbf{E}$ in (7.39), for which we have, due to (2.5)(a), (7.41)(a) and (b), and (7.42),

$$\alpha_0 \mathbf{i}^{\mathrm{T}}\mathbf{E} = v_0 q' E'_x + i'_y (E'_y + \mu v_0 H'_z) + i'_z (E'_z - \mu v_0 H'_y) . \qquad (7.45)$$

Integrating (7.45) over V, as before, the first term to the right in (7.45) leads to an integral as in the right-hand side of (7.43) and thus vanishes. Determining the contributions by the other terms amounts to evaluating the integrals

$$\int_{V'} (i'_y E'_y + i'_z E'_z) \mathrm{d}V' \quad \text{and} \quad \int_{V'} (i'_y H'_z - i'_z H'_y) \mathrm{d}V' .$$

The first one of these vanishes due to (D.12)(b) and (c) together with (D.13)(b) and (c), while the second one is zero due to (D.12)(b) and (c) together with (D.14)(b) and (c).

7.2.3 Comments

The movement of an electron Pa in a shell of an atom also occurs, in a sense, under the influence of an external electrostatic field, i.e., the one created by the charge of the nucleus. Due to the strong forces involved one must expect, however, that the field of which Pa consists will be quite torn apart and thus correspondingly spread out. We will not examine this issue in the present text but want to make at least a few related remarks.

In the conventional elementary approach to wave mechanics one uses expressions that involve, in particular, a unique frequency, Ω, a unique wave number, K, and the kinetic energy W_k. Using the de Broglie relation and equating W_k with $\hbar\Omega$, the velocity of Pa is then found to be equal to $\mathrm{d}\Omega/\mathrm{d}K$, and that derivative is therefore interpreted as a group velocity. As follows from the discussion in Appendix E3, such an interpretation is not permitted. From the point of view adopted in this text, however, Ω is simply a nominal frequency, say the appropriately defined centre of a non-vanishing frequency band, K the correspondingly

defined nominal wavelength, and, as follows from the discussion in the last paragraph of Subsection 6.9, the kinetic energy of an electron is given by $W_k = \hbar\Omega/2$. This way, the velocity of Pa is found to be given by Ω/K, and no conflict arises with a correct use of the group-velocity concept. This agrees with a multidimensional (sufficient in practice: two-dimensional) Fourier analysis of a uniformly travelling particle that is definitely not point-like but consists of a distributed field and thus gives naturally rise to non-vanishing ranges of wavelength and frequency.

8. Concluding remarks

1. Starting from *Maxwell's equations in vacuum* (the only ones considered in this paper) and based on physical reasoning involving relativistic principles, such concepts as *field velocity* $\mathbf{v}$, *rest field*, *rest energy*, *rest charge density* and *rest current density* of an electromagnetic (EM) field have been mathematically defined. By a thorough analysis of a variety of their properties, the relevance and consistency of these concepts, in particular that of field velocity, have been confirmed. The prime interest of the paper concerns *autonomous* EM fields, i.e. fields that are *self-sustaining* in the sense that all charges and currents are properties, not sources of the field.

2. Using those concepts, the known equations that involve momentum and field energy densities and are a consequence of Maxwell's equations have been remodelled using rigorous mathematical deductions. The resulting equations exhibit remarkable analogies with fundamental equations of fluid dynamics. They lend themselves to a consistent mechanistic interpretation, which justifies designating them *as flow equations* of the EM field. These become particularly attractive if we assume the EM field to be *basal*, i.e. to be such that the rest current density vanishes everywhere in the given reference frame. It appears reasonable to assume that any autonomous EM field is basal in some reference frame. As the flow equations clearly show, an EM field possesses *inertia* and thus, equivalently, *mass*.

3. These results make it highly plausible to consider EM particles such as electrons and photons as having an inner structure that is still fully described by Maxwell's equations in vacuum, thus down to the smallest dimensions. From a mechanical point of view, that inner structure may be considered to be that of a fluid, say that of an *EM fluid*. We must then

clearly distinguish between three levels of observation: At the *primary* or *basic level*, the fine details of the behaviour are relevant, in particular such important aspects as the flow of charge and momentum densities. The behaviour at this basic level is like in a mechanical fluid. In particular, changes of the energy density are caused by two entirely different mechanisms: transportation by convection (with velocity $\mathbf{v}$) and contributions due to the work done by the forces acting in the field. At the *secondary level*, both these mechanisms are considered to be combined into a single overall energy *migration*. The result is a pure flow of the energy density with an equivalent, overall velocity $\mathbf{v}_c$ that is in fact equal to what may be called the *classical energy velocity*. For small velocities we have, $\mathbf{v}_c = 2\mathbf{v}$, while $\mathbf{v}_c = \mathbf{v}$ at the speed of light. At the *tertiary level*, finally, the movement of an *EM particle* as a whole is observed, thus when ignoring its inner structure and registering only the overall movement visible from the outside.

4. As the mathematical expressions derived from Maxwell's equations show, the detailed behaviour of the structure at the basic level is governed by rules that are better compatible with the *alternative relativistic dynamics* that has become known in recent years. On the contrary, at the secondary level, some of the mathematical results, although obtained in the same rigorous way from Maxwell's theory as for the basic level, correspond to laws known from *classical relativistic dynamics*. The values of the momentum density $\mathbf{j}$ and the field-energy density w are exactly the same at both levels of observation, but the crucial difference is that between $\mathbf{v}$ and $\mathbf{v}_c$, in terms of which both $\mathbf{j}$ and w can indeed be expressed. Convection terms, however, thus terms representing flow of charge, momentum, and energy, can even then be properly interpreted only by means of the alternative theory.

5. The issue addressed in item 3 is strengthened by examining two specific field configurations. The first one is reminiscent of an *electron*. Indeed, the concept of a flowing field has lent itself to examining properties of a *rotating EM field* that is *localized*, i.e. contained in a volume of small effective size. It is described by a set of nonlinear partial differential equations (PDEs) that have been rigorously derived from Maxwell's equations. Provided the equations obtained for such a field have indeed a stable solution, two quantities can be defined that correspond essentially to the known *fine structure constant* and the magnetic *moment*. The

former is a purely mathematical number because its evaluation, which would involve solving three nonlinear PDEs and subsequently calculating two integrals, is independent of any physical property. An analytical solution of these PDEs is difficult and has not been found, at least not so far, while a numerical solution faces some difficulties and has not yet been attempted.

6. The second specific field is reminiscent of a *photon*. Since a photon must also be contained in a volume of small size it cannot be modelled by an EM field in which the charge density q and the current density $\mathbf{i}$ vanish everywhere, but the effective (total) charge must of course be strictly zero. The field thus obtained as a rigorous solution of Maxwell's equations possesses all properties one could expect to be associated with a photon; these are:
a) It propagates along a *straight line* in a single direction, with the *speed of light*.
b) The field is t*ransversal* and exhibits effective *polarization* (linear, circular).
c) The *energy* W is *proportional* to the nominal *frequency* Ω.
d) Its *momentum* is equal to W / c
e) It has *zero rest energy* and zero *rest mass*.
f) The *total charge* of the field is *zero*.
g) Its *magnetic moment* is zero.
h) It exhibits the known *relativistic Doppler* effect.
i) While the charges are *oscillating*, the *total positive* charge as well as the *total negative* charge remain individually *constant*. This could be of interest for understanding the *annihilation* of an electron and a positron as well as the corresponding *pair production*.
j) Its spread in *position* and *momentum* satisfies the classical *uncertainty relation*.
k) The field is concentrated in a small volume, thus *localized* and therefore like a *particle*.
l) It is like a general *modulated signal*, although with suppressed *carrier*, the carrier frequency corresponding to the *nominal frequency* Ω.
m) In a dispersive medium the *carrier* travels with phase velocity and the *energy* with *group velocity*.

These results strongly support the view that the field that has been described is indeed a valid *model* of a *photon*. In particular, it is simulta-

neously *wave and particle* and thus offers a natural interpretation of the *wave-particle duality*. A key to this is to consider a wave not simply as a pure sinusoid but as a modulated signal like in communications, although of a general amplitude and phase/frequency modulated type [28-30], the frequency of the (suppressed) carrier corresponding to the nominal frequency of the wave.

7. Since the movement of a particle is intimately tight to that of its energy (second level of observation), it becomes understandable why classical relativistic dynamics is the one that governs the movement of an EM particle as a whole, i.e. the behaviour at the third level of observation (cf. item 3). This is supported by results obtained in Section 7 from observing an electron-like field distribution in a second reference frame and by examining its movement in an external uniform field, electric or non-electric (for instance, gravitational). This explains why all known experiments with EM particles appear to be in agreement with the predictions of classical relativistic dynamics, contrary to what one might expect after taking a first look at the role of the alternative theory at the basic level.

Both the classical and the alternative relativistic dynamics are thus of relevance for arriving at an acceptable understanding. Each of these theories possesses its individual elegance, but it can be shown that any other attempt to develop a dynamics theory consistent with Einstein's kinematics soon becomes so unwieldy that it cannot possibly have the simplicity and flexibility any concept with broad and deep physical significance must possess. In any case, the distinction between classical and alternative relativistic dynamics is very helpful for finding meaningful interpretations of results such as the flow equations of an EM fluid etc. Nevertheless, such a distinction is not really necessary for justifying results such as those summarized under items 5 and 6. These results follow indeed rigorously from Maxwell's equations and assume thus nothing but the strict validity of these equations.

8. Some further reaching questions arise: As has just been explained, there exist strong arguments in favour of interpreting photons and electrons as condensed fields held together essentially by their own EM forces. Could something like this hold also for *other elementary particles*, surely with some other field forces involved? Could thus quantum

jumps involve highly *complex dynamic phenomena*, somewhat like what is known to be the case in digital electronics, or even far more so?

In computers, digital communications, digital control etc., in particular in the meanwhile ubiquitous internet, a gigantic number of jumps is indeed taking place every second in flip-flops, memory devices etc. all over the world. For planning all this and for assuring reliable operation, the existence of dynamic phenomena behind the jumps may be totally ignored by systems designers (except for the restrictions these phenomena impose on the operating speed), and this holds in fact for the entire field of computer science. Yet, in order to ensure proper operation of the many highly complex devices they are designing, IC experts must be very much aware of the intricate phenomena that take place in the myriad of individual circuits. If quantum jumps involve reconfigurations of fields (obviously in all three position coordinates), their dynamics would be even far more complex than that in ICs, which would contribute to explaining why many aspects in particle physics etc. can classically be reasonably assessed only by statistical approaches.

Furthermore, if other strong forces are present in the neighbourhood of a particle, its field configuration could become quite spread out or even torn apart (possibly only temporarily), in which case it would indeed be impossible to assign meaningful values to location and speed. This could be the case for the electrons in the shells of an atom and could also explain such odd phenomena as observed in two-slit experiments. Present quantum theory would in a sense concern only the tips of the icebergs, ignoring the underlying dynamical phenomena in the four-dimensional systems actually at stake.

On the other hand, most EM fields in the cosmos might in fact be *vagabonding* rather than be compressed into particles, and thus be undetectable on earth (dark matter?). We may even assume that the entire space is penetrated by an EM field that is extremely faint almost everywhere but is highly condensed in isolated small volumes. In particular, electrons (and positrons) would appear as *vortices* in the omnipresent *EM sea*, which would agree with an existing concept that elementary particles are in fact turbulence phenomena [37]. Such field condensations, however, cannot occur anyplace, but only in the neighbourhood of areas in which the charge density is non-vanishing, or better, where the divergence of the electric field is non-vanishing.

9. More generally, one may raise the question whether the only thing real in the cosmos are the fields. Elementary particles would then be nothing but condensations in the fields, in particular vortices. All the rest would be hierarchically ordered coagulations and agglomerations of, say, vortices. Like in the EM case, one would expect that condensations into elementary particles can take place only in the neighbourhood of domains in which some critical property of the field is non-vanishing.

10. Related to the present concern is also the question of *group delay* (*group velocity*). It plays a fundamental role in communications engineering, and its significance there is well understood [28-30]. It would therefore not be surprising that the importance of group velocity in physics is closely akin to that in communications, which in turn would be far easier to understand if there were some truth behind the ideas just discussed. An immediate conclusion would be that monochromatic waves and, more generally, monochromatic phenomena cannot have any true physical meaning. In full agreement with these observations, the photon-like field (wave packet) discussed under Item 6 has precisely the properties needed for our present arguments to be applicable. This is further enhanced by the role played in physics and engineering by so-called δ-functions and the systematic physical interpretation that can be given to these in communications [29] and thus, likely, also in physics itself.

11. In short: Could it be that some deeper physical insight can be gained by looking beyond the formalism by which the laws of quantum physics and also those of relativity theory are presently usually justified? The issue touches the conflict that exists between what is called realism and locality in quantum physics [38-41] and that opposed Einstein to other leading physicists of his time (Bohr, Heisenberg etc.). While the viewpoint of the latter is now widely accepted, Bell's inequalities [25], which indeed appear to confirm that conviction, appear to rest on assumptions (at least implicitly) that are not satisfied by the rather complex structures of the electron and photon models described in Section 5. Hence, the results of the present paper lend some support to Einstein's point of view. In any case, within the present theory there exists full compatibility between relativity theory and the quantum principle.

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Appendix A: Reminders

A1. Some vector and matrix properties

Throughout this paper we use familiar algebraic vector/matrix notation, a superscript T meaning transposition. Unless otherwise stated, vectors are 3-dimensional, thus

$$\mathbf{a} = (a_x, a_y, a_z)^{\mathrm{T}} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

and we employ a notation according to which

$$a = \pm |\mathbf{a}|, \quad a^2 = \mathbf{a}^{\mathrm{T}}\mathbf{a}, \quad |\mathbf{a}| \geq 0.$$

In this text, two vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be *parallel* if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, which includes the possibilities $\mathbf{a} = \mathbf{0}$ and/or $\mathbf{b} = \mathbf{0}$. If either $\mathbf{a} = k\mathbf{b}$ or $\mathbf{b} = k\mathbf{a}$, where k is a scalar, they are called *co-parallel* if $k > 0$ and *anti-parallel* if $k < 0$.

For vectors $\mathbf{a}$ and $\mathbf{b}$ in general one finds by simply expanding and rearranging,

$$(a_y b_z - a_z b_y)^2 = a^2 b^2 - (\mathbf{a}^{\mathrm{T}}\mathbf{b})^2 - a^2 b_x^2 - b^2 a_x^2 + 2a_x b_x \mathbf{a}^{\mathrm{T}}\mathbf{b},$$

$$(a_z b_x - a_x b_z)(a_y b_z - a_z b_y) = -a_x a_y b^2 - b_x b_y a^2 + (a_x b_y + a_y b_x)\mathbf{a}^{\mathrm{T}}\mathbf{b}.$$

Further expressions follow by circular permutation, leading altogether to

$$(\mathbf{a}\times\mathbf{b})(\mathbf{a}\times\mathbf{b})^{\mathrm{T}} = [a^2b^2 - (\mathbf{a}^{\mathrm{T}}\mathbf{b})^2]\mathbf{1} - (\mathbf{a}\mathbf{a}^{\mathrm{T}})b^2 - (\mathbf{b}\mathbf{b}^{\mathrm{T}})a^2 + (\mathbf{a}^{\mathrm{T}}\mathbf{b})(\mathbf{a}\mathbf{b}^{\mathrm{T}} + \mathbf{b}\mathbf{a}^{\mathrm{T}}) \tag{A.1}$$

where $\mathbf{1}$ is the unit matrix (of order 3) and $\times$ stands for the cross product. This relation, although not widely used, is quite helpful for our purpose.

Recall also the relations and implications for up to four vectors of the above type:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}^{\mathrm{T}}\mathbf{c})\mathbf{b} - (\mathbf{a}^{\mathrm{T}}\mathbf{b})\mathbf{c}, \tag{A.2}$$

$$\mathbf{a}^{\mathrm{T}}(\mathbf{b}\times\mathbf{c}) = \mathbf{b}^{\mathrm{T}}(\mathbf{c}\times\mathbf{a}) = \mathbf{c}^{\mathrm{T}}(\mathbf{a}\times\mathbf{b}), \tag{A.3}$$

$$(\mathbf{a}\times\mathbf{b})^{\mathrm{T}}(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}^{\mathrm{T}}\mathbf{c})(\mathbf{b}^{\mathrm{T}}\mathbf{d}) - (\mathbf{a}^{\mathrm{T}}\mathbf{d})(\mathbf{b}^{\mathrm{T}}\mathbf{c}), \tag{A.4}$$

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c}) = \mathbf{a}\mathbf{a}^{\mathrm{T}}(\mathbf{b}\times\mathbf{c}). \tag{A.5}$$

For arbitrary n-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$, where n is any integer ≥ 1, we have

$$\det(\mathbf{1} + \mathbf{a}\mathbf{b}^{\mathrm{T}}) = 1 + \mathbf{a}^{\mathrm{T}}\mathbf{b}, \tag{A.6}$$

$$(\mathbf{1} + \mathbf{a}\mathbf{b}^{\mathrm{T}})^{-1} = \mathbf{1} - \frac{1}{1 + \mathbf{a}^{\mathrm{T}}\mathbf{b}}\mathbf{a}\mathbf{b}^{\mathrm{T}}. \tag{A.7}$$

The following results are from vector analysis. A familiar one is

$$\nabla^{\mathrm{T}}(\mathbf{a}\times\mathbf{b}) = \mathbf{b}^{\mathrm{T}}(\nabla\times\mathbf{a}) - \mathbf{a}^{\mathrm{T}}(\nabla\times\mathbf{b}). \tag{A.8}$$

Of the next two ones, the first is known for instance in fluid dynamics and the second is a generalization of the first. They hold in Cartesian coordinates:

$$\mathbf{a}\times(\nabla\times\mathbf{a}) + [\nabla^{\mathrm{T}}(\mathbf{a}\mathbf{a}^{\mathrm{T}})]^{\mathrm{T}} = \mathbf{a}\nabla^{\mathrm{T}}\mathbf{a} + \tfrac{1}{2}\nabla a^2 \tag{A.9}$$

$$\mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a}) + [\nabla^{\mathrm{T}}(\mathbf{a}\mathbf{b}^{\mathrm{T}} + \mathbf{b}\mathbf{a}^{\mathrm{T}})]^{\mathrm{T}} = \mathbf{a}\nabla^{\mathrm{T}}\mathbf{b} + \mathbf{b}\nabla^{\mathrm{T}}\mathbf{a} + \nabla(\mathbf{a}^{\mathrm{T}}\mathbf{b}) \tag{A.10}$$

The fourth one holds in fact not only for ∇ but for any n-dimensional differential operator $\mathbf{D} = (\mathrm{D}_1, ..., \mathrm{D}_n)^{\mathrm{T}}$, D_1 to D_n being scalar (partial) differential operators, any $n\times m$ matrix $\mathbf{A}$, and any matrix $\mathbf{B}$ with m rows:

$$\mathbf{D}^{\mathrm{T}}(\mathbf{A}\mathbf{B}) = (\mathbf{D}^{\mathrm{T}}\mathbf{A})\mathbf{B} + (\mathbf{A}^{\mathrm{T}}\mathbf{D})^{\mathrm{T}}\mathbf{B}. \tag{A.11}$$

A2. Spherical coordinates

In order to be consistent with the special role we are assigning (in line with an old tradition [1]) to the $x-\text{axis}$ ((cf. 2.4)), we consider spherical coordinates r,θ,φ in terms of which the Cartesian coordinates are given by

$$x = r\cos\theta, \quad y = r\sin\theta\cos\varphi, \quad z = r\sin\theta\sin\varphi, \tag{A.12}$$

where $r \geq 0, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi$.

Writing in general $\operatorname{div}\mathbf{a}$ and $\operatorname{rot}\mathbf{a}$ rather than the Cartesian forms $\nabla^{\mathrm{T}}\mathbf{a}$ and $\nabla\times\mathbf{a}$, we have in spherical coordinates r,θ,φ, thus with $\mathbf{a} = (a_r, a_\theta, a_\varphi)^{\mathrm{T}}$,

$$\operatorname{div}\mathbf{a} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(a_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial a_\varphi}{\partial\varphi}$$

$$= \frac{\partial a_r}{\partial r} + \frac{1}{r}\frac{\partial a_\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial a_\varphi}{\partial\varphi} + \frac{2a_r}{r} + \frac{\cot\theta}{r}a_\theta,$$

$$\operatorname{rot}\mathbf{a} = (\operatorname{rot}_r\mathbf{a}, \operatorname{rot}_\theta\mathbf{a}, \operatorname{rot}_\varphi\mathbf{a})^{\mathrm{T}}, \quad \operatorname{rot}_r\mathbf{a} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(a_\varphi\sin\theta) - \frac{\partial a_\theta}{\partial\varphi}\right),$$

$$\operatorname{rot}_\theta\mathbf{a} = \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial a_r}{\partial\varphi} - \frac{\partial}{\partial r}(ra_\varphi)\right), \quad \operatorname{rot}_\varphi\mathbf{a} = \frac{1}{r}\left(\frac{\partial}{\partial r}(ra_\theta) - \frac{\partial a_r}{\partial\theta}\right),$$

$$\mathrm{d}V = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi.$$

Consider any point P with spherical coordinates r (arbitrary), θ, and φ, let $\mathbf{a}$ be a vector located at P, and let a_x, a_y, and a_z be its Cartesian components. We have

$$\left.\begin{aligned} a_x &= a_r\cos\theta - a_\theta\sin\theta, \\ a_y &= a_r\sin\theta\cos\varphi + a_\theta\cos\theta\cos\varphi - a_\varphi\sin\varphi, \\ a_z &= a_r\sin\theta\sin\varphi + a_\theta\cos\theta\sin\varphi + a_\varphi\cos\varphi. \end{aligned}\right\} \tag{A.13}$$

Appendix B: Outlines of proofs of some results in main body

B1. Proof of (2.60)

Sufficiency is easily proved by substituting (2.60) in (2.58) and (2.59), taking into account (2.62), and using (A.3) and (A.4). Let us thus concen-

trate on necessity. The requirement for $\mathbf{S}_+$ to be co-parallel with $\mathbf{u}$, $\mathbf{S}_-$ to be anti-parallel with $\mathbf{u}$, and the two superposed fields to be individually locally planar (condition (2.59)) can equivalently be expressed by

$$Z\mathbf{H}_+ = \mathbf{u}\times\mathbf{E}_+, \quad Z\mathbf{H}_- = -\mathbf{u}\times\mathbf{E}_-.$$

Due to (2.58)(a) we must thus also have

$$Z\mathbf{H} - \mathbf{u}\times\mathbf{E}_+ = Z(\mathbf{H}-\mathbf{H}_+) = Z\mathbf{H}_- = -\mathbf{u}\times\mathbf{E}_- = \mathbf{u}\times(\mathbf{E}_+ - \mathbf{E}),$$

therefore $2\mathbf{u}\times\mathbf{E}_+ = Z\mathbf{H} + \mathbf{u}\times\mathbf{E}$, and hence, after cross-multiplying from the left by $\mathbf{u}$,

$$2\mathbf{u}\times(\mathbf{u}\times\mathbf{E}_+) = Z\mathbf{u}\times\mathbf{H} + \mathbf{u}\times(\mathbf{u}\times\mathbf{E}).$$

Making use of (A.2), this yields the expression for $\mathbf{E}_+$ in (2.60)(a) and then also (2.60)(c) to (d).

B2. Proof of (3.15)

Since

$$\alpha_0\beta'_x = \beta'_x + (\alpha_0 - 1)\beta'_x = \beta'_x - \frac{\beta_0^2}{1+\alpha_0}\beta'_x$$

and due to (2.3), we can consecutively write for the vector shown between parentheses in the left-hand side of (3.11),

$$\begin{pmatrix}\alpha_0\beta'_x\\ \beta'_y\\ \beta'_z\end{pmatrix} = \boldsymbol{\beta}' + (\alpha_0-1)\beta'_x\begin{pmatrix}1\\0\\0\end{pmatrix} = \boldsymbol{\beta}' - \frac{1}{1+\alpha_0}(\boldsymbol{\beta}_0^{\mathrm{T}}\boldsymbol{\beta}')\boldsymbol{\beta}_0$$

$$= (\mathbf{1} - \frac{1}{1+\alpha_0}\boldsymbol{\beta}_0\boldsymbol{\beta}_0^{\mathrm{T}})\boldsymbol{\beta}'.$$

Hence, (3.11) can be written in the general form (3.15)(a), where the right-hand side follows easily from that of (3.11) together with (2.3). Furthermore, applying (A.7), we have

$$(\mathbf{1} - \frac{1}{1+\alpha_0}\boldsymbol{\beta}_0\boldsymbol{\beta}_0^{\mathrm{T}})^{-1} = \mathbf{1} + \frac{1}{(1+\alpha_0)(1-\frac{\beta_0^2}{1+\alpha_0})}\boldsymbol{\beta}_0\boldsymbol{\beta}_0^{\mathrm{T}}$$

$$= \mathbf{1} + \frac{1}{(1+\alpha_0)\alpha_0}\boldsymbol{\beta}_0\boldsymbol{\beta}_0^{\mathrm{T}},$$

which immediately leads to (3.15)(b). Finally, (3.15) remains unchanged if $\boldsymbol{\beta}$, $\boldsymbol{\beta}_0$, $\boldsymbol{\beta}'$ and $\boldsymbol{\gamma}$ are replaced by $\mathbf{U}\boldsymbol{\beta}$, $\mathbf{U}\boldsymbol{\beta}_0$, $\mathbf{U}\boldsymbol{\beta}'$ and $\mathbf{U}\boldsymbol{\gamma}$, where $\mathbf{U}$ is an arbitrary orthogonal matrix.

B3. Proof of (3.38)

Let RF_0 the rest reference frame associated with the point P under consideration in RF. If we assume $|\mathbf{i}_0 / q_0| < c$ and apply to RF_0 the result of Subsection 3.4 (cf. (3.31)), we are led to a reference frame RF_0' in which the rest current and the field velocity are both equal to zero. Due to the group property, the Lorentz transformation from RF to RF' can equivalently be replaced by the following three consecutive transformations: first from RF to RF_0, then from RF_0 to RF_0', and finally from RF_0' to RF'. Hence, as shown in Subsection 3.5 (cf. (3.36)(a)), the rest current $\mathbf{i}_0'$ is of the form of (3.38)(a).

The scalars i_0' and q_0' in (3.38) are still unknown and will be determined next. Due to (2.22), (3.21), (3.37), and (3.38)(a) we can first write

$$\boldsymbol{\beta}^{\mathrm{T}}\mathbf{i}_0 = \boldsymbol{\beta}'^{\mathrm{T}}\mathbf{i}_0' = 0\,;$$

then, applying (2.57) to RF and RF',

$$\left.\begin{aligned} q_0 &= \alpha q, \quad \mathbf{i} = \boldsymbol{\gamma} i_0 + q\mathbf{v}, && \text{(a)} \\ q_0' &= \alpha' q', \quad \mathbf{i}' = \boldsymbol{\gamma}' i_0' + q'\mathbf{v}'; && \text{(a)} \end{aligned}\right\} \tag{B.1}$$

pre-multiplying (B.1)(a) by $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$, respectively, and taking into account (3.19),

$$\left.\begin{aligned} i_0 &= \boldsymbol{\gamma}^{\mathrm{T}}\mathbf{i}_0 = \boldsymbol{\gamma}^{\mathrm{T}}\mathbf{i}, && \text{(a)} \\ i_0' &= \boldsymbol{\gamma}'^{\mathrm{T}}\mathbf{i}_0' = \boldsymbol{\gamma}'^{\mathrm{T}}\mathbf{i}' = \mathbf{i}'^{\mathrm{T}}\mathbf{M}\boldsymbol{\gamma}; && \text{(b)} \end{aligned}\right\} \tag{B.2}$$

and finally, pre-multiplying $\mathbf{i}$ in (B.1)(a) by $\boldsymbol{\beta}^{\mathrm{T}}$,

$$\boldsymbol{\beta}^{\mathrm{T}}\mathbf{i} = cq\beta^2 = cq(1-\alpha^2)\,. \tag{B.3}$$

In view of (2.7)(a) to (c) and (3.20) we therefore obtain from (B.2)(b),

$$i_0' = \frac{\alpha'}{\alpha\alpha_0}\Big[\Big(i_x - \beta_0(cq - \boldsymbol{\beta}^{\mathrm{T}}\mathbf{i} + \beta_x i_x)\Big)\gamma_x + (1-\beta_0\beta_x)(\boldsymbol{\gamma}^{\mathrm{T}}\mathbf{i} - \gamma_x i_x)\Big]$$

whence (3.38)(b) follows by making use of (B.2)(a) and (B.3).

By means of (2.7)(d) and the expression for i_x that can be extracted from (B.1)(a) we find

$$\alpha_0 cq' = cq(1-\beta_0\beta_x) - \beta_0 i_{0x}\,.$$

From there, (3.38)(c) follows easily by means of (B.1)(b) and (3.8).

B4. **An equivalence proof for Subsection** 4.1

Substitute first (4.2) in (4.4) and then (4.3) to (4.5) in (4.1)(a). Recalling the definition (2.9), the result can be written in the form

$$\frac{1}{c^2}\mathbf{E}\times \mathrm{D}_t\mathbf{H} + \varepsilon\left[\frac{1}{2}\nabla E^2 - \left(\nabla^{\mathrm{T}}(\mathbf{E}\mathbf{E}^{\mathrm{T}})\right)^{\mathrm{T}} + \mathbf{E}\nabla^{\mathrm{T}}\mathbf{E}\right]$$
$$+\mu\left[\frac{1}{2}\nabla H^2 - \left(\nabla^{\mathrm{T}}(\mathbf{H}\mathbf{H}^{\mathrm{T}})\right)^{\mathrm{T}} - \mathbf{H}\times(\nabla\times\mathbf{H})\right] = \mathbf{0}.$$

Due to (A.9) this yields

$$\varepsilon\mathbf{E}\times(\mu \mathrm{D}_t\mathbf{H} + \nabla\times\mathbf{E}) - \mu\mathbf{H}\nabla^{\mathrm{T}}\mathbf{H} = \mathbf{0}\,. \tag{B.4}$$

Similarly, using (4.2)(b) as well as (A.8) and referring again to (2.9), we deduce from (4.1)(b),

$$\mathbf{H}^{\mathrm{T}}(\mu \mathrm{D}_t\mathbf{H} + \nabla\times\mathbf{E}) = 0\,. \tag{B.5}$$

We then form the cross product of $\mathbf{H}$ and (B.4) and make use of (A.2) and (B. 5). This yields,

$$(\mu \mathrm{D}_t\mathbf{H} + \nabla\times\mathbf{E})\mathbf{E}^{\mathrm{T}}\mathbf{H} = \mathbf{0}\,.$$

Thus, if $\mathbf{E}^{\mathrm{T}}\mathbf{H} \neq 0$, (2.1)(b) holds and then, in view of (B.4), also (2.2)(b). Furthermore, even if $\mathbf{E}^{\mathrm{T}}\mathbf{H} = 0$ at the point P under consideration the result still holds by continuity if in every neighbourhood of P there exist points where $\mathbf{E}^{\mathrm{T}}\mathbf{H} \neq 0$. The issue is trivial for (2.1)(a) and (2.2)(a).

B5. Proof of (4.16)

For outlining the proof of (4.16) we first pre-multiply by $\mathbf{v}^{\mathrm{T}}$ the individual terms in the left-hand side of (4.14)(a). The result for the last one can be evaluated by means of (4.15). For the other three terms we can write, in short,

$$\left.\begin{aligned}\mathbf{v}^{\mathrm{T}}\frac{\partial\mathbf{j}}{\partial t} &= \boldsymbol{\beta}^{\mathrm{T}}\frac{\partial}{\partial t}\left(\frac{2w_0}{\alpha^2}\boldsymbol{\beta}\right) = \frac{\partial}{\partial t}\left(\frac{2w_0}{\alpha^2}\beta^2\right) - \frac{w_0}{\alpha^2}\frac{\partial\beta^2}{\partial t}\\ &= \frac{\partial}{\partial t}\left(\frac{w_0}{\alpha^2}\beta^2\right) + \beta^2\frac{\partial}{\partial t}\left(\frac{w_0}{\alpha^2}\right),\end{aligned}\right\} \tag{B.6}$$

$$\left.\begin{aligned}\mathbf{v}^{\mathrm{T}}[\nabla^{\mathrm{T}}(\mathbf{v}\mathbf{j}^{\mathrm{T}})]^{\mathrm{T}} &= (\nabla^{\mathrm{T}}(\frac{2w_0}{\alpha^2}\mathbf{v}\boldsymbol{\beta}^{\mathrm{T}}))\boldsymbol{\beta}\\ &= \nabla^{\mathrm{T}}(\frac{\beta^2}{\alpha^2}2w_0\mathbf{v}) - \frac{w_0}{\alpha^2}(\mathbf{v}^{\mathrm{T}}\nabla)\beta^2\\ &= \nabla^{\mathrm{T}}(\frac{2w_0}{\alpha^2}\mathbf{v}) - 2\nabla^{\mathrm{T}}(w_0\mathbf{v}) + \frac{w_0}{\alpha^2}(\mathbf{v}^{\mathrm{T}}\nabla)\alpha^2\\ &= \nabla^{\mathrm{T}}(\frac{2w_0}{\alpha^2}\mathbf{v}) - \nabla^{\mathrm{T}}(w_0\mathbf{v}) - \alpha^2\nabla^{\mathrm{T}}(\frac{w_0}{\alpha^2}\mathbf{v}),\end{aligned}\right\} \quad \text{(B.7)}$$

$$\mathbf{v}^{\mathrm{T}}(\nabla^{\mathrm{T}}\mathbf{T}_0)^{\mathrm{T}} = (\nabla^{\mathrm{T}}\mathbf{T}_0)\mathbf{v} = \nabla^{\mathrm{T}}(\mathbf{T}_0\mathbf{v}) - (\mathbf{T}_0^{\mathrm{T}}\nabla)^{\mathrm{T}}\mathbf{v}. \quad \text{(B.8)}$$

The proof of (4.16) is then completed by indeed pre-multiplying (4.14)(a) by $\mathbf{v}^{\mathrm{T}}$, subtracting the result from (4.14)(b), making use of (B.6) to (B.8) and (4.15), combining suitable terms, and dividing by α^2.

Appendix C: Alternative relativistic dynamics

C1. Summary of results on alternative relativistic dynamics

The basic results of the alternative relativistic dynamics have been obtained by several distinct approaches [13-15]. They are based on the same four *essentials* as the classical theory, i.e.,

1. Einstein's criticism of Newtonian time and space,
2. Einstein's postulates,
3. the Lorentz transformation,
4. the Einsteinian kinematics.

In order to present some of the key results let $\mathbf{F}$ be the force acting on a point mass of rest mass M_0 and travelling with velocity $\mathbf{v}$. According to the *alternative approach* to relativistic dynamics [13-15] the classical expression

$$\mathbf{F}_c = \mathrm{D}(M_c\mathbf{v}), \qquad M_c = \frac{M_0}{\alpha} \quad \text{(C.1)}$$

should be replaced by

$$\mathbf{F} = \sqrt{M}\mathrm{D}(\sqrt{M}\mathbf{v}) = \frac{1}{2}(\mathrm{D}(M\mathbf{v}) + M\mathrm{D}\mathbf{v}), \quad M = \frac{M_0}{\alpha^2} \quad \text{(C.2)}$$

where in both cases,

$$\mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}, \quad \alpha = \sqrt{1-\beta^2}, \quad \beta = \frac{v}{c}, \quad v^2 = \mathbf{v}^{\mathrm{T}}\mathbf{v}, \quad \text{(C.3)}$$

both expressions for $\mathbf{F}$ in (C.2) being strictly equivalent. From (C.2) one finds for the kinetic energy

$$W_k = \frac{\beta^2}{\alpha^2} W_0 = \frac{1}{2} M v^2, \qquad W_0 = \frac{1}{2} M_0 c^2, \tag{C.4}$$

while W_0 rather than the classical $W_{0c} = M_0 c^2$ is the preferred choice for the rest energy and thus

$$W = W_0 + W_k = \frac{W_0}{\alpha^2} = \frac{1}{2} M c^2 \tag{C.5}$$

that for the total energy.

The expression for $\mathbf{F}$ given by (C.2) is identical to that of the so-called *world force* or *Minkowski force*. The 4-tuple formed by $\mathbf{F}$ and $\mathbf{v}^{\mathrm{T}}\mathbf{F}/c$ is a true four-vector. Due to this, $\mathbf{F}$ also appears in modern presentations of relativity theory but only as a simplifying mathematical tool, thus without being given any specific physical meaning. Hence, while momentum and reduced energy (i.e., energy divided by c) form a four-vector in classical relativity, the same is true for force and reduced power in the alternative theory.

Adopting (C.2) requires Newton's third law, which is taken over unquestioned in classical relativity, thus in the form $\mathbf{F}_{c1} = -\mathbf{F}_{c2}$, to be replaced by

$$\alpha_1 \mathbf{F}_1 = -\alpha_2 \mathbf{F}_2, \quad \alpha_i = \sqrt{1-\beta_i^2}, \quad \beta_i = \frac{v_i}{c}, \quad v_i^2 = \mathbf{v}_i^{\mathrm{T}} \mathbf{v}_i, \quad i = 1, 2. \tag{C.6}$$

These equations refer to two interacting objects numbered 1 and 2, $\mathbf{v}_i$ being the velocity of object i and $\mathbf{F}_i$ the force exerted on object i by the other one. Let then the momentum $\mathbf{J}$ be defined by

$$\mathbf{J} = \frac{M_0}{\alpha} \mathbf{v}, \tag{C.7}$$

i.e., in exactly the same way for the alternative as for the classical theory. The definitions (C.2) and (C.2) can then be written

$$\mathbf{F}_c = \mathrm{D}\mathbf{J}, \quad \mathbf{F} = \frac{1}{\alpha} \mathrm{D}\mathbf{J} = \frac{1}{\alpha} \mathbf{F}_c. \tag{C.8}$$

As a consequence of this and (C.6), conservation of momentum holds in exactly the same way as in the classical theory. This implies in particular that in the equations expressing conservation of momentum the expressions for the forces acting upon a point mass are the same as those in the

absence of movement, thus those that would hold if the velocity of the point mass were zero.

In the case of a field, the analysis presented for instance in [15] is incomplete and thus does not properly cover the subject. In particular, only the action of external forces had been taken into considerations, thus neither the specifics concerning internal surface and volume forces in an autonomous field nor the possibility of energy transmission by other mechanisms (cf. the term $\alpha \mathbf{i}_0^{\mathrm{T}} \mathbf{E}_0$ in (4.15)). It appears best, therefore, to take a more pragmatic attitude and to concentrate on ad hoc interpretations of strictly valid results, making sure of course that such interpretations do not violate formerly established principles. We proceed accordingly in Section 4.

C2. Comparison with classical derivations

An interesting issue is to compare key results of the alternative theory with original derivations published by Einstein. In order to obtain an expression for the kinetic energy W_k he had considered a particle of rest mass ΔM_0 and charge ΔQ that is exposed to an electrostatic field of constant strength E acting solely in the x-direction, is travelling with velocity v, and is at rest at $x = 0$. Using rules that follow strictly from the Lorentz transformation, one has,

$$E\Delta Q = \frac{\Delta M_0}{\alpha^3} \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}, \tag{C.9}$$

as derived in [1,4]. Let F be the actual force acting on the particle and $F_c = E\Delta Q$ its classical expression. Einstein takes it for granted that $F = F_c$ and thus obtains for the kinetic energy the classical expression

$$W_{kc} = \int_0^x F \,\mathrm{d}x = \int_0^x E\Delta Q \,\mathrm{d}x = \Delta M_0 \int_0^x \frac{v}{\alpha^3} \mathrm{d}v = c^2 \Delta M_0 (\frac{1}{\alpha} - 1).$$

This is fully compatible with the result of Subsection 4.3. As we have seen there, the movement of a particle as an entity of its own is a phenomenon at the tertiary level of observation and thus follows the laws of classical relativistic dynamics.

This would be different if we could single out from inside of an EM fluid an elementary charge, say $\Delta Q = q\Delta V$, and subject it to an external field E as just considered. This would be a phenomenon at the primary

level and would therefore be governed by the laws of the alternative relativistic dynamics, thus with $F = F_c / \alpha$. Consequently, we would obtain

$$W_k = \int_0^x F\,\mathrm{d}x = \int_0^x \frac{F_c}{\alpha}\,\mathrm{d}x = \int_0^x \frac{E\Delta Q}{\alpha}\,\mathrm{d}x = \Delta M_0 \int_0^v \frac{v}{\alpha^4}\,\mathrm{d}v = \frac{v^2 \Delta M_0}{2\alpha^2},$$

which corresponds precisely to the result given by (C.4).

The situation is more subtle concerning the relation between mass and energy. Einstein himself [1, 2, 4] uses a thought experiment (Gedanken-experiment) that, in a sense, may be said to be over-idealized. It becomes more realistic if one replaces the uniform planar field considered by Einstein by the compact planar field described in Subsection 5.3. If one then still assumes, like tacitly done by Einstein, that the loss of mass suffered by the emitting object is exclusively due to the energy gained by the emitted radiation, one arrives again at his famous conclusion, in agreement with what we have seen for the energy properties at the secondary and tertiary levels. In particular, since the object emitting the radiation in Einstein's Gedankenexperiment is composed of particles, its mass decrease must follow the rules of classical relativistic dynamics.

C3. Flow equations of a basal EM field: mechanistic derivation of (4.22)

We restrict ourselves to basal EM fields. It is instructive to examine how a flow equation such as (4.22) can be derived by essentially mechanistic arguments, i.e., under the assumption that the alternative relativistic dynamics is the one that has to be used at the basic level. We postulate conservation of momentum, i.e., the existence of an equation such as (4.6)(a) in which the momentum density $\mathbf{j}$ appears in the first two additive terms and is itself proportional to the velocity $\mathbf{v}$, which in turn is pre-multiplying $\mathbf{j}^{\mathrm{T}}$ in the second additive term. In other words, we start by assuming that $\mathbf{v}$ and a scalar m_i, called in this text inertial mass density, exist such that $\mathbf{j} = m_i \mathbf{v}$ (cf. (4.31)(a)) holds. Let $m_0 = m_i\big|_{\mathbf{v}=\mathbf{0}}$ be the associated rest mass density, and let us determine an appropriate relationship between these quantities.

For doing this, we consider an elementary volume ΔV that contains in its interior the specific point P under consideration in RF. Let ΔV_0 be the corresponding elementary volume in the rest reference frame

RF_0, and $\Delta\mathbf{J}$ the elementary momentum in ΔV, with $\Delta V = \alpha\Delta V_0$, α being related to $\mathbf{v}$ as always in this text (cf. (2.11)(b) and (2.17)). The elementary rest mass in ΔV_0 is given by $\Delta M_0 = m_0\Delta V_0$ where, according to the alternative relativistic dynamics, $m_0 = 2w_0/c^2$, w_0 being the rest energy density. Since P is moving with velocity $\mathbf{v}$ with respect to P_0, which is the point in RF_0 that corresponds to P, we can write $\Delta\mathbf{J} = (\Delta M_0/\alpha)\mathbf{v}$ and thus for the momentum density,

$$\mathbf{j} = \frac{\Delta\mathbf{J}}{\Delta V} = \frac{\Delta M_0}{\alpha\Delta V}\mathbf{v} = \frac{m_0\Delta V_0}{\alpha\Delta V}\mathbf{v} = \frac{2w_0}{\alpha^2c^2}\mathbf{v},$$

which is precisely as required by (4.7). This way the first two additive terms in (4.22) can be expressed by means of $\mathbf{v}$ and w_0. The specific expression (2.24) for w_0 cannot, of course, be obtained by mechanistic arguments, and the same kind of observation holds for the forces.

The next two terms in (4.22) are indeed due to the surface and volume forces acting in the field. The volume force density $\mathbf{f}_0$ has been found to be given by (4.20)(b). As mentioned there, this is the value to which $\mathbf{f}_c$ reduces for $\mathbf{v} = \mathbf{0}$. The fourth term in (4.22) is thus explained if we assume that in an equation expressing conservation of momentum, with momentum density $\mathbf{j}$ appearing as in (4.22), the force densities are those that would exist at P if the field were there at rest, i.e., if at P we had $\mathbf{v} = \mathbf{0}$ (as is compatible with properties of the alternative relativistic dynamics mentioned in C1). For properly understanding this assumption note that it amounts to stating that the volume force density in the conservation equation is equal to the value of $\mathbf{f}_c$ at P_0, but this is *not* the same thing as saying that $\mathbf{f}_c$ is obtained by determining it at P_0 and then transforming the result from RF_0 to RF.

Although less immediately visible, the third term in (4.22), which takes into account the surface forces, can be interpreted by essentially the same type of arguments. Accordingly, there must exist, for $\mathbf{v} = \mathbf{0}$, a tensor $\mathbf{T}_0$ that occurs in (4.22) in the way shown there. For justifying the expression given by (4.9), let us offer the following argument: We observe first that the criticism of the classical tensor $\mathbf{T}_c$ mentioned in Subsection 4.1 (cf. the paragraph comprising (4.6)) does not apply to

electrostatic and magnetostatic fields, where the classical derivations are indeed unobjectionable. In both these cases we have $\mathbf{v} = \mathbf{0}$, i.e., $w = w_0$. Furthermore, if the field is electrostatic we have $\mathbf{E} = \mathbf{E}_0$ and $\mathbf{H} = \mathbf{H}_0 = \mathbf{0}$, and if it is magnetostatic, $\mathbf{E} = \mathbf{E}_0 = 0$, $\mathbf{H} = \mathbf{H}_0$, in both cases thus altogether $\mathbf{T}_c = \mathbf{T}_0$, with $\mathbf{T}_c$ given by (4.5) and $\mathbf{T}_0$ by (4.9). Hence, if we apply the classical derivation we find indeed for the surface forces the tensor $\mathbf{T}_0$ defined in (4.9) and thus for their contribution to (4.22) the third term in that equation. It therefore appears at least plausible to assume that the result obtained for an electrostatic or a magnetostatic field holds, more generally, at all P where $\mathbf{v} = \mathbf{0}$.

Appendix D: Symmetry requirements for a rotating field

D1. Requirements in spherical coordinates

We list hereafter some properties that an acceptable solution of (5.6) to (5.10) must, or at least, might exhibit:

1. For reasons of symmetry, E_r and E_θ must vanish for $r = 0$.
2. For large values of r, the electric field must approach that of a point charge. Hence, E_r must then decrease like $1/r^2$ and must become independent of θ.
3. For large values of r, E_θ must decrease at the rate of $1/r^4$. This can indeed be shown to be the case for the component E_θ of the electric field produced by two equal point charges located symmetrically on the $\theta = 0$ axis.
4. For small values of r the magnetic field must essentially behave like that of a circular current loop near $r = 0$, for which corresponding results can, for instance, be found in [9, Section 5.5]. In particular, neither H_r nor H_θ may vanish for $r = 0$, and we must have
$$\lim_{r\to 0} H_r\big|_{\theta=0} = -\lim_{r\to 0} H_\theta\big|_{\theta=\pi/2}.$$
5. For the same reason as for small values of r, H_r and H_θ must, for large values of r, decrease like $1/r^3$. More precisely, for large values of r, H_r must behave like $\cos\theta/r^3$ and H_θ like $\sin\theta/r^3$.

6. For reasons of symmetry, E_θ must vanish everywhere for $\theta = 0$, $\theta = \pi/2$, and $\theta = \pi$. This is achieved if E_θ contains the factor $\sin\theta\cos\theta$.
7. Again for reasons of symmetry, we must have $H_r = 0$ for all r if $\theta = \pi/2$ but not if $\theta = 0$ or $\theta = \pi$, while $H_\theta = 0$ for all r if $\theta = 0$ and $\theta = \pi$ but not if $\theta = \pi/2$. These properties are all achieved if H_r contains the factor $\cos\theta$ and H_θ the factor $\sin\theta$.
8. For reasons of continuity and physical appropriateness, the field velocity v must vanish for all θ if $r = 0$ or $r = \infty$, and furthermore for all points on the axis of rotation, i.e., for all r if $\theta = 0$. It must not vanish, however, for $\theta = \pi/2$ and all r except if $r = 0$ or $r = \infty$. The requirements for θ are achieved if v contains the factor $\sin\theta$. That factor appears automatically if we express v in terms of the angular velocity ω according to $v = r\omega\sin\theta$.
9. In addition to containing the factors in θ we have been considering, E_r, E_θ, H_r, H_θ, q, and v may still be dependent on θ. Again for reasons of symmetry, this must be such that no changes occur if θ is replaced by either $\pi - \theta$ or $-\theta$. This is achieved if the remaining dependence on θ is in terms of

$$s = \sin^2\theta, \tag{D.1}$$

in which case we may write

$$\frac{\partial}{\partial\theta} = 2\sin\theta\cos\theta\frac{\partial}{\partial s}. \tag{D.2}$$

In view of some of the properties we have addressed it might appear tempting to try solving (5.6) to (5.10) by requiring $E_{0\theta} = H_{0\theta} = 0$. This would satisfy (5.10) and, in view of (5.9a) and (5.9b), would be equivalent to

$$E_r = \frac{E_{0r}}{\alpha}, \quad H_r = \frac{H_{0r}}{\alpha}, \quad E_\theta = -\mu v H_r, \quad H_\theta = \varepsilon v E_r.$$

A solution on that basis, however, would not be acceptable because the last one of these equations would not be compatible with the requirements $\lim_{r\to 0} H_\theta \neq 0$ and $\lim_{r\to 0} E_r = \lim_{r\to 0} v = 0$. Hence, a more general approach is needed.

In view of the various requirements mentioned above, it is appropriate to define $g_r, g_\theta, h_r, h_\theta, b$ and s by means of

$$\left.\begin{aligned} &\sqrt{\varepsilon E_r} = g_r, \quad \sqrt{\varepsilon} E_\theta = g_\theta \sin\theta\cos\theta, \quad \beta = b\sin\theta, \\ &\sqrt{\mu} H_r = h_r \cos\theta, \quad \sqrt{\mu} H_\theta = h_\theta \sin\theta. \end{aligned}\right\} \tag{D.3}$$

together with (D.1) and (D.2). This way, (5.6) to (5.8) can be written

$$r\frac{\partial g_\theta}{\partial r} + g_\theta - 2\frac{\partial g_r}{\partial s} = 0, \tag{D.4}$$

$$r\frac{\partial h_r}{\partial r} + 2h_r + 2s\frac{\partial h_\theta}{\partial s} + 2h_\theta = 0, \tag{D.5}$$

$$\begin{aligned} b\Big(r\frac{\partial g_r}{\partial r} + 2g_r + 2s(1-s)\frac{\partial g_\theta}{\partial s} + (2-3s)g_\theta\Big) \\ = r\frac{\partial h_\theta}{\partial r} + h_\theta - 2(1-s)\frac{\partial h_r}{\partial s} + h_r, \end{aligned} \tag{D.6}$$

while we deduce from (5.12) and (5.13),

$$\frac{2bw}{1+sb^2} = g_r h_\theta - (1-s) g_\theta h_r, \tag{D.7}$$

$$w = g_r^2 + s(1-s)g_\theta^2 + (1-s)h_r^2 + sh_\theta^2. \tag{D.8}$$

The quantities $g_r, g_\theta, h_r, h_\theta,$ and b, clearly, are functions of r and s.

D2. Some symmetry properties in Cartesian coordinates

For applications in Section 7 the details presented above are irrelevant. Only some more general properties that strictly follow from the circular and equatorial assumptions are needed, and it is convenient to have them available in Cartesian coordinates. For achieving this, let ε_x, ε_y, and ε_z be auxiliary parameters defined by

$$\varepsilon_x = \pm 1, \quad \varepsilon_{y,} = \pm 1, \quad \varepsilon_z \pm 1, \tag{D.9}$$

where the three double-signs can be chosen independently of one another. This way a single position (x, y, z) gives rise to an octuplet $(\varepsilon_x x, \varepsilon_y y, \varepsilon_z z)$ of symmetrically placed positions, one in each octant of the three-dimensional space. For all (x, y, z) and all permitted choices of ε_x, ε_y, and ε_z the wanted symmetries can then be expressed as follows:

$$q(\varepsilon_x x, \varepsilon_y y, \varepsilon_z z) = q(x, y, z), \tag{D.10}$$

$$\left.\begin{aligned} &v_x(x,y,z)=0, \quad v_y(\varepsilon_x x,\varepsilon_y y,\varepsilon_z z)=\varepsilon_z v_y(x,y,z),\\ &v_z(\varepsilon_x x,\varepsilon_y y,\varepsilon_z z)=\varepsilon_y v_z(x,y,z), \end{aligned}\right\} \tag{D.11}$$

$$\left.\begin{aligned} &i_x(x,y,z)=0, \quad i_y(\varepsilon_x x,\varepsilon_y y,\varepsilon_z z)=\varepsilon_z i_y(x,y,z), \quad \text{(a), (b)}\\ &i_z(\varepsilon_x x,\varepsilon_y y,\varepsilon_z z)=\varepsilon_y i_z(x,y,z), \quad \text{(c)} \end{aligned}\right\} \tag{D.12}$$

$$\left.\begin{aligned} &E_x(\varepsilon_x x,\varepsilon_y y,\varepsilon_z z)=\varepsilon_x E_x(x,y,z), \quad \text{(a)}\\ &E_y(\varepsilon_x x,\varepsilon_y y,\varepsilon_z z)=\varepsilon_y E_y(x,y,z), \quad \text{(b)}\\ &E_z(\varepsilon_x x,\varepsilon_y y,\varepsilon_z z)=\varepsilon_z E_z(x,y,z), \quad \text{(c)} \end{aligned}\right\} \tag{D.13}$$

$$\left.\begin{aligned} &H_x(\varepsilon_x x,\varepsilon_y y,\varepsilon_z z)=H_x(x,y,z), \quad \text{(a)}\\ &H_y(\varepsilon_x x,\varepsilon_y y,\varepsilon_z z)=\varepsilon_x\varepsilon_y H_y(x,y,z), \quad \text{(b)}\\ &H_z(\varepsilon_x x,\varepsilon_y y,\varepsilon_z z)=\varepsilon_x\varepsilon_z H_z(x,y,z), \quad \text{(c)} \end{aligned}\right\} \tag{D.14}$$

Full circular symmetry implies the additional identities

$$q(x,z,y)=q(x,y,z),$$

$$\left.\begin{aligned} &v_y(x,z,y)=-v_z(x,y,z), \quad i_z(x,z,y)=-i_y(x,y,z), \quad \text{(a),(b)}\end{aligned}\right\} \tag{D.15}$$

$$\left.\begin{aligned} &E_x(x,z,y)=E_x(x,y,z), \quad H_x(x,z,y)=H_x(x,y,z), \quad \text{(a),(b)}\end{aligned}\right\} \tag{D.16}$$

$$\left.\begin{aligned} &E_y(x,z,y)=E_z(x,y,z), \quad H_y(x,z,y)=H_z(x,y,z). \quad \text{(a),(b)}\end{aligned}\right\} \tag{D.17}$$

This way the number of positions of interest is even further increased to a total of 16, a pair of complementary positions appearing now in every octant. Further useful expressions can be obtained by combining (A.13) with (D.3).

Appendix E: Analytic signals and related results

E1. Analytic signal

Let $f(t)$ be a real or complex-valued function of t and $F(j\omega)$ its (complex-valued) *Fourier transform*, j being the imaginary unit. Using a notation that is convenient in communications (where the related concepts of complex frequency and Laplace transform play an important role) [28, 29], we have,

$$\left.\begin{aligned} F(j\omega) &= \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\,\mathrm{d}t, \quad \text{(a)} \\ f(t) &= \mathcal{F}^{-1}\{F(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t}\,\mathrm{d}\omega. \quad \text{(b)} \end{aligned}\right\} \tag{E.1}$$

More specifically, if $f(t)$ is a *real function* of t we have $F(-j\omega) = F^*(j\omega)$ where the asterisk denotes complex conjugation. It is helpful to associate with a real $f(t)$ a so-called *analytic signal* $f_+(t)$, which is a complex-valued function of the real variable t [28, 29, 31 - 34, 43 - 45]. It can easiest be defined by requiring for its Fourier transform

$$F_+(j\omega) = \mathcal{F}\{f_+(t)\} = \begin{cases} 2F(j\omega) & \text{for} \quad \omega > 0 \\ F(0) & \text{for} \quad \omega = 0 \\ 0 & \text{for} \quad \omega < 0, \end{cases} \tag{E.2}$$

and (E.1) then yields,

$$\left.\begin{aligned} f(t) &= \mathrm{Re}\, f_+(t), \quad \text{(a)} \\ f_+(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_+(j\omega)\, e^{j\omega t}\,\mathrm{d}\omega = \frac{1}{\pi}\int_{0}^{\infty} F(j\omega) e^{j\omega t}\,\mathrm{d}\omega, \quad \text{(b)} \end{aligned}\right\} \tag{E.3}$$

where Re is the operator that selects the real part of the subsequent expression. Obviously, even if $F(j\omega)$ is continuous everywhere, as we always assume, $F_+(j\omega)$ is discontinuous at $\omega = 0$ unless $F(0) = 0$. In practical applications that latter condition is usually strictly satisfied, in idealized theoretical context (for instance Gauss function in the next subsection) at least with a very high degree of accuracy.

We are interested in particular in real functions $f(t)$ of band-pass type. Assume thus that for some $\omega_0 > 0$ and some $\Delta\omega$ with $0 < \Delta\omega < 2\omega_0$ we have

$$F(j\omega) = 0 \quad \text{for} \quad |\omega| > \omega_0 + \tfrac{1}{2}\Delta\omega \text{ and for } |\omega| < \omega_0 - \tfrac{1}{2}\Delta\omega. \tag{E.4}$$

The *complex amplitude* of $f(t)$ with respect to ω_0, $\hat{f}(t)$, or equivalently, its Fourier transform $\hat{F} = \mathcal{F}\{\hat{f}\}$, are then defined by

$$\left.f_+(t) = e^{j\omega_0 t}\hat{f}(t), \quad \hat{F}(j\omega) = F_+(j\omega_0 + j\omega). \quad \text{(a),(b)}\right\}. \tag{E.5}$$

In view of (E.2) and (E.4), (E.5) implies $\hat{F}(j\omega) = 0$ for $|\omega| > \frac{1}{2}\Delta\omega$. Hence, if $\Delta\omega << \omega_0$, $\hat{f}(t)$ is slowly varying compared to $e^{j\omega_0 t}$.

In applications such as in Subsection 5.3 there is more than one variable involved. In order to avoid confusion it may then be appropriate to add a subscript in order to specify which one of the independent variables is intended, thus to write $\mathcal{F}_t$ and $\mathcal{F}_t^{-1}$ instead of simply $\mathcal{F}$ and $\mathcal{F}^{-1}$ as in (E.1) etc.

E2. Uncertainty relation

While for signals in communications theory the precise formulation of the equivalent of Heisenberg's *uncertainty relation* is often discussed only for signals of low-pass type centred at $t_0 = 0$ [29, 34, 35], meaningful results for signals $f(t)$ of band-pass type are easiest obtained by replacing $F(j\omega)$ by $F_+(j\omega)$ and thus, in order to assure a proper transformation pair, $f(t)$ by $f_+(t)$ [31, 43, 44]. The question sometimes raised whether such a replacement doesn't falsify the outcome [43, 45], however, is irrelevant in our case. Firstly, indeed, since no strict physical argument exists for imposing a specific precise definition of duration and bandwidth, the crucial point is to select definitions that are, on the one hand, physically meaningful and easy to interpret and, on the other, sufficiently easy to handle mathematically. Secondly, as will be briefly discussed below, the replacement of f by f_+ does not affect the outcome for band-pass signals f such as those that are of interest in our context. Although the present derivation of the wanted inequality (E.15) is based on arguments essentially similar to those used by Gabor [31] in his extension of the Pauli-Weyl [42] approach, all salient steps are briefly included for sake of completeness.

Convenient measures for the *duration* Δt and the *bandwidth* $\Delta\omega$ are now given by

$$\left.\begin{aligned} \left(\frac{\Delta t}{2}\right)^2 &= \frac{\int_{-\infty}^{\infty}(t-t_0)^2\,|f_+(t)|^2\mathrm{d}t}{\int_{-\infty}^{\infty}|f_+(t)|^2\mathrm{d}t}, \qquad \text{(a)} \\ \left(\frac{\Delta\omega}{2}\right)^2 &= \frac{\int_{-\infty}^{\infty}(\omega-\omega_0)^2\,|F_+(j\omega)|^2\mathrm{d}\omega}{\int_{-\infty}^{\infty}|F_+(j\omega)|^2\mathrm{d}\omega}, \quad \text{(b)} \end{aligned}\right\} \qquad \text{(E.6)}$$

where the constants t_0 and ω_0 should assume the role of time centre and frequency centre and where we want $\Delta t/2$ and $\Delta\omega/2$ to be in a sense the "radii of gyration" of the "mass distributions" $|f_+(t)|^2$ and $|F_+(j\omega)|^2$, respectively. For achieving this, t_0 and ω_0 must be such that (E.6) defines the smallest possible values of both Δt and $\Delta\omega$. This is precisely the case if

$$\left.\begin{aligned} t_0 &= \frac{\int_{-\infty}^{\infty} t\,|f_+(t)|^2\,\mathrm{d}t}{\int_{-\infty}^{\infty} |f_+(t)|^2\,\mathrm{d}t}, \quad \omega_0 = \frac{\int_{-\infty}^{\infty} \omega\,|F_+(j\omega)|^2\,\mathrm{d}\omega}{\int_{-\infty}^{\infty} |F_+(j\omega)|^2\,\mathrm{d}\omega}, \quad \text{(a),(b)} \end{aligned}\right\} \qquad \text{(E.7)}$$

i.e., if t_0 and ω_0 are the respective "centres of gravity", or *centroids*, with $\Delta t/2$ and $\Delta\omega/2$ then becoming the root mean square (RMS) deviations from the mean values of t and ω, respectively.

The function $\tilde{f}(t) = f(t_0 + t)$ coincides with $f(t)$ except for a shift that brings the time centroid to the position $t = 0$. Its Fourier transform is given by

$$\tilde{F}(j\omega) = \mathcal{F}\{f(t_0 + t)\} = e^{j\omega t_0} F(j\omega),$$

and that of the analytic signal associated with $\tilde{f}(t)$ thus by $\tilde{F}_+(j\omega) = e^{j\omega t_0} F_+(j\omega)$ (cf. (E.2)). We therefore have,

$$\tilde{f}_+(t) = \mathcal{F}^{-1}\{e^{j\omega t_0} F_+(j\omega)\} = f_+(t_0 + t).$$

For ease of notation let us denote the associated complex amplitude, defined with respect to ω_0, simply by ψ instead of $\hat{\tilde{f}}$, as a strict adherence to the adopted notation would require (cf. (E.5)). Hence, we can write,

$$\left.\begin{aligned} \psi(t) &= e^{-j\omega_0 t} f_+(t + t_0), && \text{(a)} \\ \Psi(\mathrm{j}\omega) &= \mathcal{F}\{\psi(t)\} = e^{j(\omega+\omega_0)t_0} F_+(j\omega + j\omega_0). && \text{(b)} \end{aligned}\right\} \qquad \text{(E.8)}$$

We then derive from (E.6),

$$\left(\frac{\Delta t}{2}\right)^2 = \frac{\int_{-\infty}^{\infty} t^2\,|f_+(t_0 + t)|^2\,\mathrm{d}t}{\int_{-\infty}^{\infty} |f_+(t_0 + t)|^2\,\mathrm{d}t} = \frac{\int_{-\infty}^{\infty} t^2\,|\psi(t)|^2\,\mathrm{d}t}{\int_{-\infty}^{\infty} |\psi(t)|^2\,\mathrm{d}t}, \qquad \text{(E.9)}$$

$$\left(\frac{\Delta\omega}{2}\right)^2 = \frac{\int_{-\infty}^{\infty}\omega^2\,|F_+(j\omega_0+j\omega)|^2\mathrm{d}\omega}{\int_{-\infty}^{\infty}|F_+(j\omega_0+j\omega)|^2\mathrm{d}\omega} = \frac{\int_{-\infty}^{\infty}|j\omega\Psi(j\omega)|^2\mathrm{d}\omega}{\int_{-\infty}^{\infty}|\Psi(j\omega)|^2\mathrm{d}\omega} = \frac{\int_{-\infty}^{\infty}|\dot\psi(t)|^2\mathrm{d}t}{\int_{-\infty}^{\infty}|\psi(t)|^2\mathrm{d}t}, \tag{E.10}$$

where the last step in (E.10) can be justified by observing that $j\omega\Psi(j\omega)$ is the Fourier transform of $\mathrm{d}\psi(t)/\mathrm{d}t = \dot\psi(t)$ and by applying Parseval's equation. The resulting expression

$$\left(\frac{\Delta t}{2}\right)^2\left(\frac{\Delta\omega}{2}\right)^2 = \frac{\int_{-\infty}^{\infty}t^2\,|\psi(t)|^2\mathrm{d}t\int_{-\infty}^{\infty}|\dot\psi(t)|^2\mathrm{d}t}{\left(\int_{-\infty}^{\infty}|\psi(t)|^2\mathrm{d}t\right)^2} \tag{E.11}$$

has now the same general structure as the one classically encountered, with ψ replacing the real-valued function f, but the function ψ is necessarily complex valued.

We can now make use of the Schwarz inequality in any of the forms resulting from

$$\left.\begin{aligned}\int_{-\infty}^{\infty}|g|^2\,\mathrm{d}t\int_{-\infty}^{\infty}|h|^2\mathrm{d}t &\geq \left(\int_{-\infty}^{\infty}|gh|\,\mathrm{d}t\right)^2\\ &\geq \left|\int_{-\infty}^{\infty}g^*h\mathrm{d}t\right|^2 \geq \left|\int_{-\infty}^{\infty}\mathrm{Re}(g^*h)\mathrm{d}t\right|^2,\end{aligned}\right\} \tag{E.12}$$

where $g = g(t)$ and $h = h(t)$ are complex-valued functions of the real variable t. In (E.12), assuming $g(t)$ not to vanish identically, all three inequalities become equalities simultaneously if and only if $h(t)+kg(t)=0\ \forall t$, where k is any real constant, the first two inequalities become equalities simultaneously if and only if $h(t)+kg(t)=0\ \forall t$, where k is any complex constant, and the first inequality becomes an equality if and only if $|h(t)|=|kg(t)|\ \forall\mathrm{t}$, where k is again any complex constant. If we choose

$$\mathrm{g}(t) = t\psi(t) \quad \text{and} \quad h(t) = \frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \dot\psi(t), \tag{E.13}$$

(E.12) yields,

$$\begin{aligned}\int_{-\infty}^{\infty} t^2 |\psi|^2 \,\mathrm{d}t \int_{-\infty}^{\infty} |\dot{\psi}|^2 \,\mathrm{d}t &\geq \int_{-\infty}^{\infty} |t\psi\dot{\psi}|^2 \,\mathrm{d}t \geq \left|\int_{-\infty}^{\infty} t\psi\dot{\psi}\mathrm{d}t\right|^2 \\ &\geq \left|\int_{-\infty}^{\infty} \mathrm{Re}(t\psi^*\dot{\psi})\mathrm{d}t\right|^2 = \frac{1}{4}\left(\int_{-\infty}^{\infty} |\psi|^2 \,\mathrm{d}t\right)^2,\end{aligned} \tag{E.14}$$

where, integrating by parts as in standard presentations, we have made use of

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{Re}(t\psi^*\dot{\psi})\mathrm{d}t &= \frac{1}{2}\int_{-\infty}^{\infty} t[\psi^*\dot{\psi} + \psi\dot{\psi}^*]\mathrm{d}t \\ &= \frac{1}{2}\int_{-\infty}^{\infty} t\mathrm{d}(\psi\psi^*) = -\frac{1}{2}\int_{-\infty}^{\infty} |\psi|^2 \mathrm{d}t\end{aligned}$$

and of the obvious assumption that $f_+(t)$ and thus $\psi(t)$ vanish sufficiently fast for $t \to \pm\infty$. Using (E.11) and exploiting the inequality between the first and the last term in (E.14), we can now immediately write the inequalities

$$\Delta t \cdot \Delta\omega \geq 2, \quad \text{i.e.,} \quad \left(\frac{\Delta t}{2}\right)\left(\frac{\Delta\omega}{2}\right) \geq \frac{1}{2}, \tag{E.15}$$

the first one holding for the measures of the spread in time, Δt, and the spread in frequency, $\Delta\omega$, the second one for the corresponding "radii of inertia" $\Delta t/2$ and $\Delta\omega/2$.

Due to (E.13), the three mentioned requirements for achieving equalities can be written, respectively:

$$\dot{\psi}(t) = -kt\psi(t), \quad \text{i.e.,} \quad \frac{\mathrm{d}}{\mathrm{d}t^2}\ln\psi(t) = -\frac{k}{2}, \quad k \in \mathbb{R}; \tag{E.16}$$

$$\left.\begin{aligned}&\dot{\psi}(t) = -kt\psi(t), \quad \text{i.e.,} \quad \frac{\mathrm{d}}{\mathrm{d}t^2}\ln\psi(t) = -\frac{k}{2}, \\ &k = k' + jk'', \quad k', k'' \in \mathbb{R};\end{aligned}\right\} \tag{E.17}$$

$$\left.\begin{aligned}&|\dot{\psi}(t)| = |kt\psi(t)| \Rightarrow \left|\frac{\mathrm{d}}{\mathrm{d}t^2}\ln\psi(t)\right| = \left|-\frac{k}{2}\right| \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t^2}\ln\psi(t) = -\frac{k}{2}e^{jb(t)}, \quad k, b(t) \in \mathbb{R},\end{aligned}\right\} \tag{E.18}$$

where the last result is obtained by observing that $\mathrm{d}\ln\psi(t)/\mathrm{d}t^2$ must be equal to a real constant times a function of modulus one, which we may write as $-k$ and $e^{jb(t)}$, respectively, $b(t)$ being an arbitrary real

function of t. Of the solutions of (E.16) to (E.18) only those are acceptable that vanish for $t \to \pm\infty$, i.e., respectively,

$$\psi(t) = Ce^{-kt^2/2}, \quad k > 0, \tag{E.19}$$

$$\psi(t) = Ce^{-(k'+jk'')t^2/2}, \quad k' > 0, \quad k'' \in \mathbb{R}, \tag{E.20}$$

$$\left.\begin{aligned} &\psi(t) = Ce^{-A(t)-jB(t)}, \quad A(t) = k\int_0^t t\cos b(t)\mathrm{d}t, \\ &B(t) = k\int_0^t t\sin b(t)\mathrm{d}t, \quad k > 0, \end{aligned}\right\} \tag{E.21}$$

where in all three cases $C = |C|e^{j\gamma}$ is an arbitrary complex constant and where b in (E.21) must be such that $e^{-A(t)}$ vanishes for $t \to \pm\infty$. We may also assume $A(0) = B(0) = 0$ since any $A(0) \neq 0$ and/or `$B(0) \neq 0$ could be absorbed in C.

Clearly, the solution given by (E.19) has the remarkable feature to be the only one to which corresponds a bound that is independent of any signal property. On the other hand, (E.21) is the solution that exhausts most completely the freedom offered by (E.12). It makes the first one of the three inequalities in (E.14) become an equality without requiring the same to hold also for the other two, and it reduces to (E.20) if $b(t)$ is an appropriate constant, say b_0, and to (E.19) if $b_0 = 0$. This justifies paying some attention to the general solution given by (E.21), for which we obtain, using (E.8)(a),

$$\left.\begin{aligned} f_+(t) &= \psi(t-t_0)e^{j\omega_0(t-t_0)} = Ce^{-A(t-t_0)-jB(t-t_0)+j(t-t_0)\omega_0} \\ &= \hat{f}(t)e^{j\omega_0 t} \end{aligned}\right\} \tag{E.22}$$

and thus in particular,

$$\left.\begin{aligned} f(t) &= \mathrm{Re}\, f_+(t) \\ &= |C|e^{-A(t-t_0)}\cos(B(t-t_0) + (t_0-t)\omega_0 - \gamma). \end{aligned}\right\} \tag{E.23}$$

For the spectral function corresponding to f_+ (cf. (E.22)) we find, using the second equality in (E.8),

$$F_+(j\omega) = \Psi(j\omega - j\omega_0)e^{j(\omega_0-\omega)t_0},$$

which can be evaluated if $\Psi(j\omega)$ is known. The integration required for this cannot in general be carried out explicitly, but for a Gauss function as in (E.19) it is again given, as is well known, by again a Gauss function, i.e., by

$$\Psi(j\omega) = \sqrt{\frac{2\pi}{k}} C e^{-\omega^2/2k}. \qquad \text{(E.24)}$$

In fact, the same expression remains valid in the more general case of (E.20) provided we choose $k = k' + jk''$ and select that square root of k for which $\mathrm{Re}\sqrt{k} > 0$, as can for instance be shown by proceeding in the following way: Adopt $z = t\sqrt{k/2} + j\omega/\sqrt{2k}$ instead of t as integration variable and the sequence of segments $-T\sqrt{k/2} + j\omega/\sqrt{2k} \to -T\sqrt{k/2} \to -T\,\mathrm{Re}\sqrt{k/2} \to T\,\mathrm{Re}\sqrt{k/2} \to T\sqrt{k/2} \to T\sqrt{k/2} + j\omega/\sqrt{2k}$ as integration path, take the limit $T \to \infty$, and observe that $(\mathrm{Re}\sqrt{k})^2 - (\mathrm{Im}\sqrt{k})^2 = \mathrm{Re}\,k = k' > 0$.

Obviously, (E.24) extends over the entire frequency range $-\infty < \omega < \infty$. It cannot therefore be expected that the resulting $F_+(j\omega)$ (cf. (E.8)) vanishes completely for $\omega < 0$, but this is irrelevant since for $\Delta\omega \ll \omega_0$, as is always the case in practice, $F_+(j\omega)$ will be totally negligible except in a small band centred at ω_0, and the same can be expected to hold in the general case of (E.21).

Let us still point out some results one obtains by substituting (E.19) to (E.21) in (E.11). Firstly, we note that we obtain from (E.20),

$$|\psi(t)|^2 = |C|^2\, e^{-2k't^2}$$

$$|\dot{\psi}(t)|^2 = |C|^2\,(k'^2 + k''^2)t^2 e^{-2k't^2} = (k'^2 + k''^2)t^2\,|\psi(t)|^2$$

and therefore after some calculation,

$$\left(\frac{\Delta t}{2}\right)\left(\frac{\Delta\omega}{2}\right) = \frac{\sqrt{k'^2 + k''^2}}{2k'} \geq \frac{1}{2}. \qquad \text{(E.25)}$$

In there, the limit $1/2$ is reached for $k'' = 0$, thus for the solution (E.19), as must be the case. Similarly, one finds for the solution (E.21),

$$\left(\frac{\Delta t}{2}\right)\left(\frac{\Delta\omega}{2}\right) = k\,\frac{\int_{-\infty}^{\infty} t^2 e^{-2A(t)}\,\mathrm{d}t}{\int_{-\infty}^{\infty} e^{-2A(t)}\,\mathrm{d}t},$$

but the required integrations cannot in general be carried out explicitly. However, (E.15) obviously remains valid, although equality can hold only in the case of (E.19), as we have seen. Remarkably, the limit $1/2$ in

(E.15) and (E.25) is independent of any property of ψ. Since this limit is actually reached if and only if ψ is as given by (E.19), none of the more general solutions can satisfy a tight bound that is equally independent of ψ but different from 1/2.

Finally, let us briefly address the issue of f_+ versus f in defining Δt. Since $\mathcal{F}\{f_+(t_0+t)\} = e^{j\omega t_0}F_+(j\omega)$, we have $\mathcal{F}\{tf_+(t_0+t)\} = j\mathrm{d}(e^{j\omega t_0}F_+(j\omega))/\mathrm{d}\omega$. Due to (E.2), $\mathrm{d}(F_+(j\omega))/\mathrm{d}\omega$ comprises the impulse $2F(0)\delta(\omega)$, which is disturbing but vanishes if $F(0)=0$, i.e., if $F_+(j\omega)$ is continuous also at $\omega = 0$. If that is the case, we derive from the first equality (E.9), making use of Parseval's theorem,

$$\left(\frac{\Delta t}{2}\right)^2 = \frac{\int_{-\infty}^{\infty} \left|\frac{\mathrm{d}}{\mathrm{d}\omega}\left(e^{j\omega t_0}F_+(j\omega)\right)\right|^2 \mathrm{d}\omega}{\int_{-\infty}^{\infty} |F_+(j\omega)|^2 \mathrm{d}\omega}$$

$$= \frac{\int_{0}^{\infty} \left|\frac{\mathrm{d}}{\mathrm{d}\omega}\left(e^{j\omega t_0}F(j\omega)\right)\right|^2 \mathrm{d}\omega}{\int_{0}^{\infty} |F(j\omega)|^2 \mathrm{d}\omega}.$$

This is exactly the same result as the one we obtain if we use f instead of f_+ in the definition of Δt in (E.6), as in fact is known [43, 44]. The requirement $F(0)=0$ simply states that the signal does not contain any zero-frequency component, which indeed is always fulfilled for usual signals of band-pass type. It is true that this is not strictly the case for the ideal solutions (E.19) to (E.21), but even then it holds with extremely good accuracy due to the narrow-band assumption.

E3. Group delay and group velocity

We first consider the transmission of a signal by a linear constant system N having one input and one output. The full behaviour of N involves not only pure forward transmission but also backward transmission coupled with reflections at both input and output. Altogether, a 2×2 scattering matrix [46] is thus needed in order to fully characterize N. Nevertheless, the overall transmission effect can be described by a single transfer function that we may represent in the form

$$e^{-\Gamma}, \quad \Gamma = A + jB, \quad \Gamma(j\omega) = A(\omega) + jB(\omega),$$

the *loss* $A(\omega)$ and the *phase* $B(\omega)$ being real functions of ω. These quantities, which depend on ω, may not be confused with the $t-$dependent quantities $A(t)$ and $B(t)$ used in Appendix E2 and defined by (E.21). Since in the present context N is necessarily passive (a consequence of conservation of energy), and if the transfer function is properly defined, we have $A(\omega) \geq 0 \; \forall \omega$. We denote by $f = f(t)$ the signal available at the input and by $g = g(t)$ the signal finally received at the output. Let f_+, $\hat{f}$, F, F_+, and $\hat{F}$ be defined as so far, and let g_+, $\hat{g}$, G, G_+, and $\hat{G}$ be the corresponding functions defined for g. We have,

$$G = e^{-\Gamma} F, \quad G_+ = e^{-\Gamma} F_+, \quad \hat{G} = e^{-\Gamma} \hat{F}. \quad \text{(a),(b),(c)} \Big\} \tag{E.26}$$

The transmission is ideal if $g(t) = f(t - t_g)$ where t_g is a constant. If there is dissipation (absorption) inside of N and/or if there are reflections at the input and output accesses, the functions $A(\omega)$ and $B(\omega)$ may cause $g(t)$ to differ substantially from $f(t)$. On the other hand, if there is neither dissipation nor reflection we have $A(\omega) = 0 \; \forall \omega$; although there is then no *amplitude distortion* there may remain a disturbing *phase distortion*.

Of special interest is the case that

$$A(\omega) = A_0 \geq 0, \quad B(\omega) = B_0 + \omega t_g, \quad \omega \in I, \quad \text{(a),(b)} \Big\} \tag{E.27}$$

where A_0, B_0, and t_g are real constants and where the interval I is the relevant range of positive frequencies. In view of that restriction to $\omega > 0$ we must be careful when applying (E.27) to (E.26). Obviously, (E.26)(b) and (c) do not cause any problem. In particular, (E.26)(b) directly yields

$$\left.\begin{aligned} g_+(t) &= e^{-A_0 - jB_0} f_+(t - t_g) = e^{-A_0} e^{j\omega_0 t - jB(\omega_0)} \hat{f}(t - t_g), \\ \hat{g}(t) &= e^{-A_0 - jB(\omega_0)} \hat{f}(t - t_g), \end{aligned}\right\} \tag{E.28}$$

$$g(t) = \operatorname{Re} g_+(t) = e^{-A_0} \operatorname{Re}\{e^{j\omega_0 (t - t_{ph}(\omega_0))} \hat{f}(t - t_g)\}, \tag{E.29}$$

where we have made use of (E.5, first equality) and (E.27), and where the *phase delay* t_{ph} is defined by

$$t_{ph}(\omega) = B(\omega) / \omega .$$

The presence of B_0 in (E.28) will usually still cause $g(t)$ to be not only delayed (and damped if $A_0 > 0$) but also distorted. In communications, this distortion does not affect the information contained in f provided proper modulation methods have been used [28, 29]. For our present application (Subsection 6.7), however, the relevant quantity is not the information content but $|\hat{f}(t)|$, thus $|\hat{g}(t)|$, in which case B_0 has no effect either. Clearly, if (E.27) strictly holds, the constant t_g satisfies

$$t_g = \frac{\mathrm{d}B(\omega)}{\mathrm{d}\omega}. \tag{E.30}$$

Since t_g is also the delay that affects the complex amplitude $\hat{f}(t)$, (E.29) explains why $\mathrm{d}B/\mathrm{d}\omega$ is often referred to as the *group delay* of the system.

Although the results we have described are rigorously valid only if (E.27) strictly holds, it is often sufficient in practice that (E.27) is satisfied with a sufficiently good approximation. The constant t_g should then be chosen in such a way that the function $B_0 + \omega t_g$ is the best possible linear approximation of $B(\omega)$ in $\omega \in I$. One is then tempted to evaluate $\mathrm{d}B/\mathrm{d}\omega$ at the centre, ω_0, of the interval I and to consider that result as the actual delay, but this may lead to substantial errors. Altogether, it should be clear that a proper interpretation of the derivative $\mathrm{d}B/\mathrm{d}\omega$ as a delay is dependent on a number of requirements that must be fulfilled with sufficient accuracy. We come back to this point after defining the velocities corresponding to t_{ph} and t_g.

Assume indeed the system N to be stretched uniformly along the x-axis, and $x = 0$ to correspond to the location of the input. Assume furthermore that there are no reflections at the input and the output, i.e., that in N pure transmission from the input to the output is taking place, thus pure forward transmission. We then have,

$$A = ax, \quad B = kx, \quad a = a(\omega), \quad k = k(\omega), \tag{E.31}$$

where k (not to be confused with the parameter k in Appendix E.2) is in fact identical to what is called the *wave number*. We may then replace (E.27) by the requirement

$$\left.\begin{array}{l} a(\omega) = a_0 \geq 0, \quad k(\omega) = k_0 + \dfrac{\omega}{v_g}, \quad \omega \in I, \quad \text{(a),(b)} \end{array}\right\} \tag{E.32}$$

where a_0, k_0, and v_g are real constants and where I is again the relevant range of positive frequencies.

In order to make clear the dependence of g on x we now use the notation $g(t,x)$ instead of simply $g(t)$, and we have $f(t) = g(t,0)$. We may then again apply (E.28) and (E.29) where, in view of (E.31) and (E.32), we now have

$$A_0 = a_0 x, \quad B_0 = k_0 x, \quad t_g = x / v_g, \quad t_{ph} = t_{ph}(\omega) = xk(\omega)/\omega .$$

For the analytic signal and the complex amplitude that are associated with $g(t,x)$, and for that signal itself, we obtain from (E.28) and (E.29),

$$g_+(t,x) = e^{-(a_0 + jk_0)x} f_+(t - \frac{x}{v_g}), \quad \hat{g}(t,x) = e^{-[a_0 + jk(\omega_0)]x} \hat{f}(t - \frac{x}{v_g}), \tag{E.33}$$

$$g(t,x) = \mathrm{Re}\, g_+(t) = e^{-a_0 x}\, \mathrm{Re}\{e^{j\omega_0[t - x/v_{ph}(\omega_0)]} \hat{f}(t - \frac{x}{v_g})\}, \tag{E.34}$$

where the *phase velocity* is defined by

$$v_{ph} = v_{ph}(\omega) = \frac{x}{t_{ph}} = \frac{\omega}{k(\omega)} \tag{E.35}$$

and where we have made use of the relation between an analytic signal and the associated complex amplitude (cf. (E.5)). More specifically, we obtain from (E.33),

$$\left.\begin{array}{l} |\, g_+(t,x)\,| = e^{-a_0 x}\, |\, f_+(t - \dfrac{x}{v_g})\,|, \quad \text{(a)} \\ |\, \hat{g}(t,x)\,| = e^{-a_0 x}\, |\, \hat{f}(t - \dfrac{x}{v_g})\,|. \quad \text{(b)} \end{array}\right\} \tag{E.36}$$

According to (E.34), $\hat{f}$ propagates with velocity v_g, and the factor $e^{j\omega_0 t}$ with velocity $v_{ph}(\omega_0)$. In addition there is also a damping effect if $a_0 > 0$ and, possibly, a distortion due to the presence of k_0 in (E.32) and thus in v_{ph} (cf. (E.35)). As (E.36) shows, however, we have pure propagation with constant velocity v_g if $a_0 = 0$ and if for the issue under ex-

amination only the modulus of the analytic signal or, equivalently, that of the complex amplitude is of relevance.

If $k(\omega)$ is indeed given by (E.32)(b), with k_0 and v_g strictly independent of ω in I, we have,

$$\left.\begin{aligned} \frac{\mathrm{d}k}{\mathrm{d}\omega} = \frac{1}{v_g}, \quad \text{thus} \quad v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}, \quad \text{(a),(b)} \end{aligned}\right\} \tag{E.37}$$

where (E.37)(b) assumes ω to be expressed in terms of k. In view of the result expressed by (E.37) the derivative $\mathrm{d}\omega(k)/\mathrm{d}k$, which in general is a function of k, or equivalently, of ω, is usually called the *group velocity*. This definition is actually adopted even if k is any given function of ω, thus even if $\mathrm{d}\omega/\mathrm{d}k$ is any function of ω, and v_g therefore itself a non-constant function of ω.

It must be emphasized, however, that this is in general only a rather formal definition, not one having a definite, universal physical significance. A proper interpretation of $\mathrm{d}\omega/\mathrm{d}k$ as a meaningful velocity is indeed again dependent on a number of requirements that must be fulfilled, as we have seen to be the case for the group delay. Unfortunately, it is common practice to ignore these requirements, or at least their full implications. This appears to be one of the reasons for the many misconceptions that exist, in communications as well as in physics, about group delay and group velocity. On the other hand, it should be mentioned that the requirement $a_0 = 0$ is not a handicap in physical applications such as the one addressed in Section 6. As we have pointed out before, we have indeed $A(\omega) = 0\ \forall\omega$, thus $a(\omega) = 0\ \forall\omega$, if there is neither reflection nor dissipation involved.

In presentations of the concepts of group delay and group velocity it is traditionally assumed, at least implicitly, that the bandwidth of the signal is narrow (which justifies breaking off the Taylor-series expansion of $B(\omega)$ after the linear term) and that the quantity of final interest is the signal envelope. Both these assumptions are neither sufficient, as follows from the above analysis, nor necessary, as is well confirmed in the field of electrical communications, thus in a discipline whose core task is the faithful transmission of the information contained in EM signals [28, 29]. In that discipline, the concepts of group delay and group velocity play

indeed a major role, and this nowhere only for narrow-band amplitude-modulated signals. They are equally important for wide-band signals (e.g. in television) and for frequency-modulated signals (where speaking about propagation of an envelope is meaningless). In any case, like we have seen for t_g, instead of defining v_g by (E.37)(b) it is more correct to define it in such a way that $1/v_g$ is the slope of an optimal linear approximation of $k(\omega)$ in, again, the relevant frequency range (cf. (E.32)(b)). Claims of having achieved superluminal signal transmission involve experiments in which, in the relevant frequency range, the loss $A(\omega)$ is large and far from being a constant, and similar observations might also hold for claims concerning the (mathematically related) tunnelling effect.

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