The nonlinear Wannier-Stark problem and resonant tunnelling

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http://mail.df.unipi.it/∼saw/index.html

Program

1. Wannier-Stark problem
   - Bloch oscillations
   - Wannier-Stark ladder and resonant tunnelling

2. Mean field-description of a Bose-Einstein condensate
   - 3D Gross-Pitaevskii equation
   - Robust observables: survival and recurrence probabilities

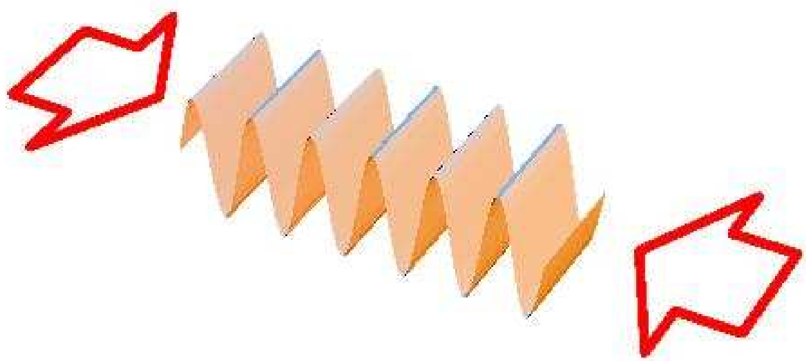
3. Resonance states?
   - Time-dependent decay rates
   - Complex-scaling method
   - Nonlinearity-induced shift of resonant tunnelling peaks

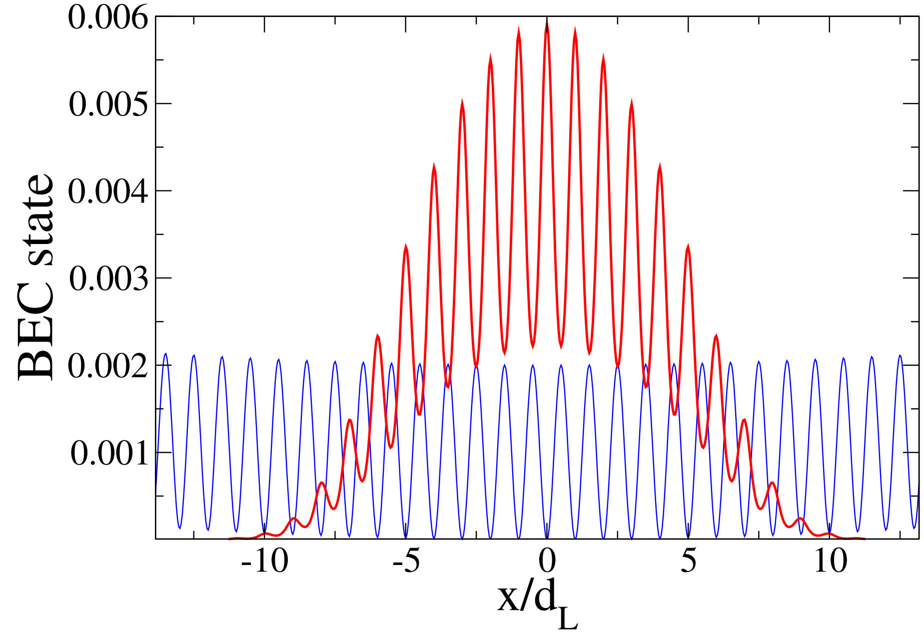
4. Conclusions

1. Wannier-Stark problem

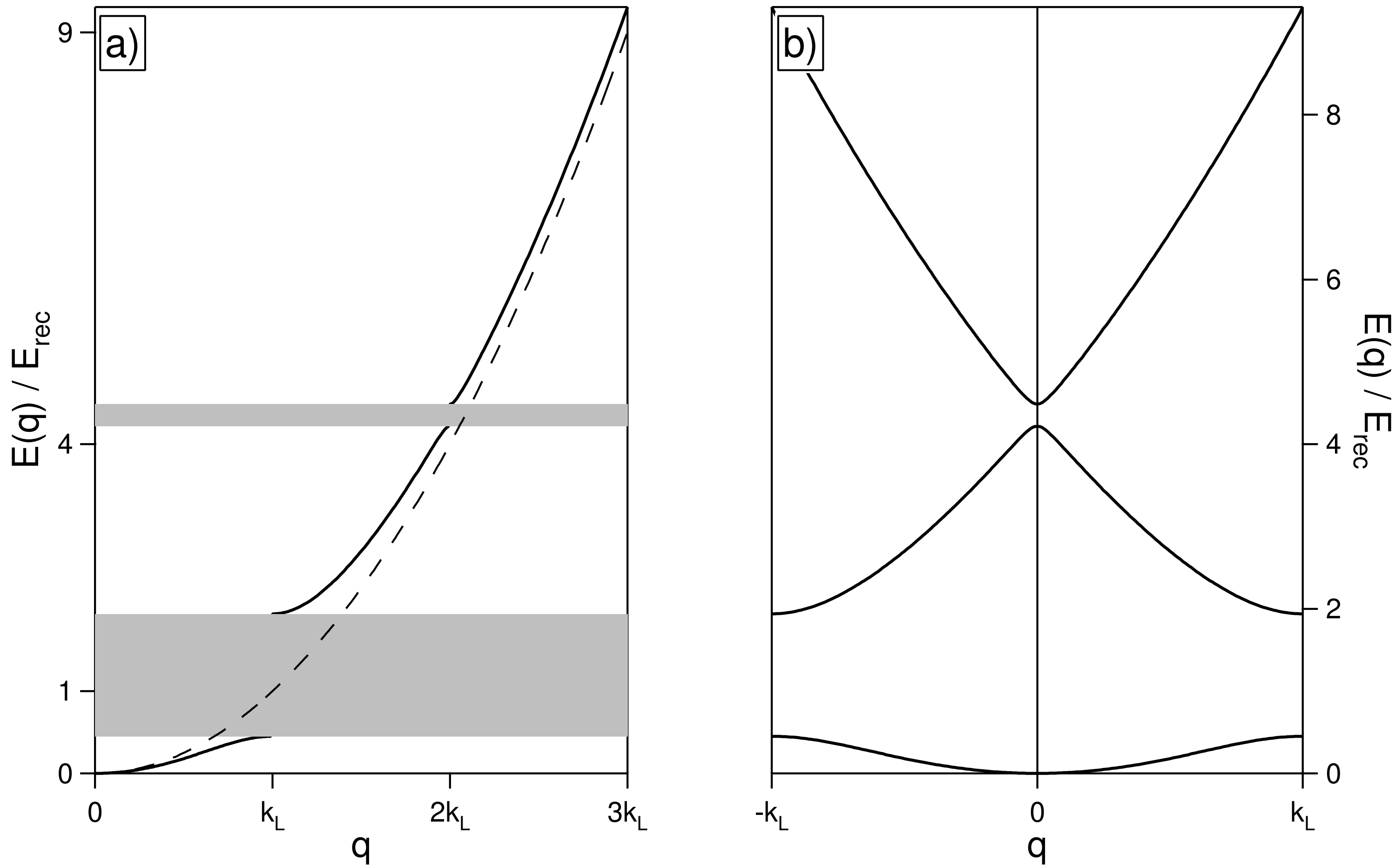
Quantum transport in a spatially periodic optical lattice:

$$\mathrm{i}\frac{\partial}{\partial t}\psi(x,t) = \left[-\frac{1}{2}\frac{\partial^2}{\partial x^2} + V_0 \sin^2(x)\right]\psi(x,t)$$



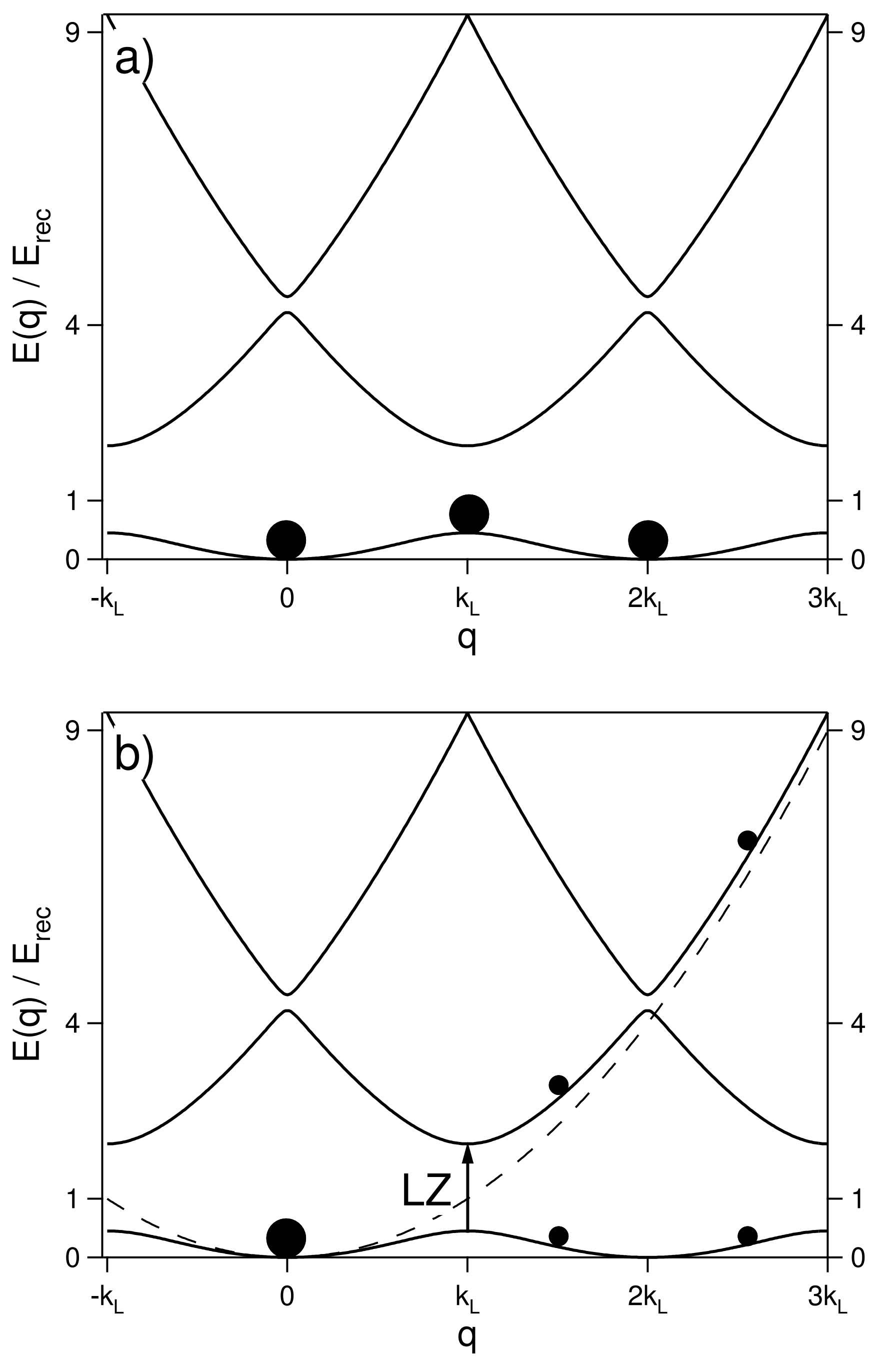


Band structure and Bloch waves without gravity/static force:

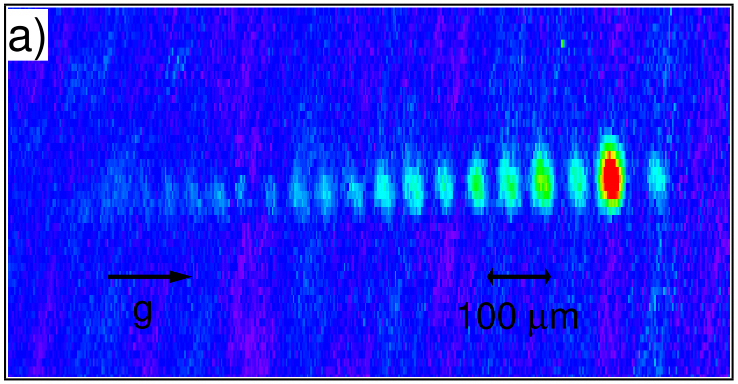


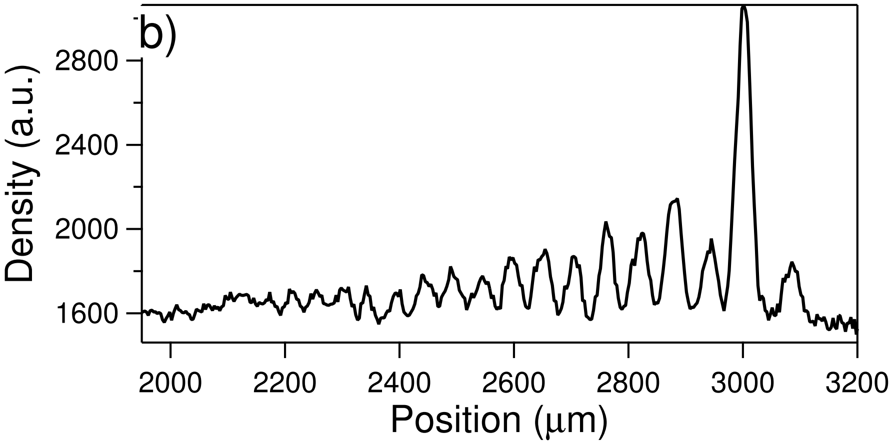
dispersion law with quasi-momentum q

Bloch oscillations



Bloch oscillations





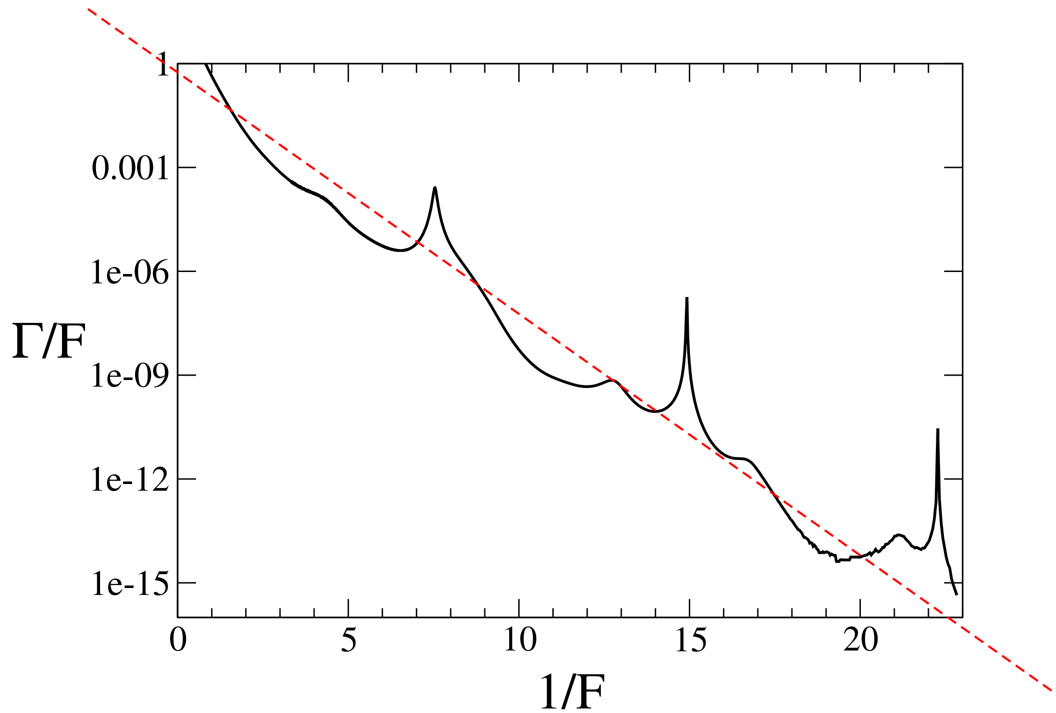
$$\mathsf{i}\,\frac{\partial}{\partial t}\psi(x,t) = \left[-\frac{1}{2}\frac{\partial^2}{\partial x^2} + V_0 \sin^2(x) + Fx\right]\psi(x,t)\ .$$

Landau-Zener tunnelling

Decay via interband tunnelling after each Bloch period $T_{\mathrm{B}} = 2\pi\hbar/d_L F$:

$$\Gamma_{\mathrm{LZ}} \sim F \exp\left(-\frac{c}{d_L F}\right)$$

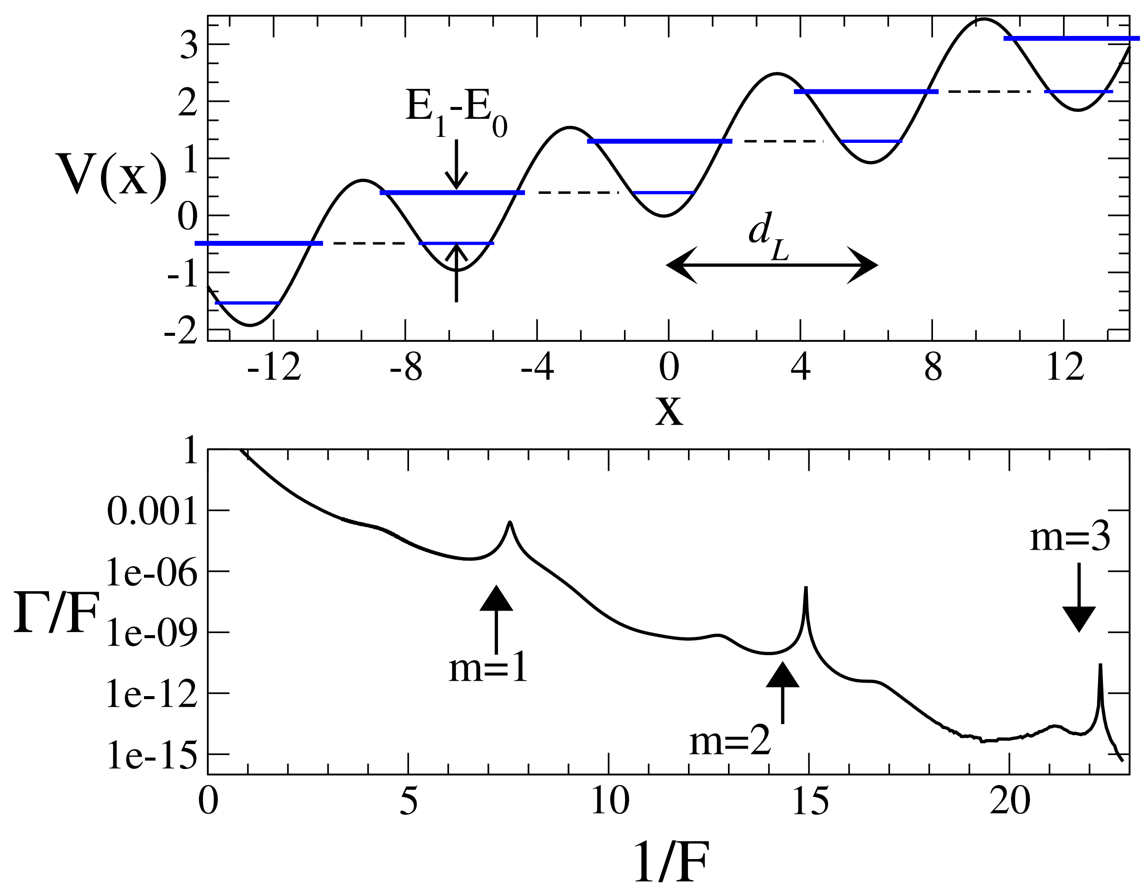
with constant $c \approx (\text{band gap})^2 \approx V_0^2$



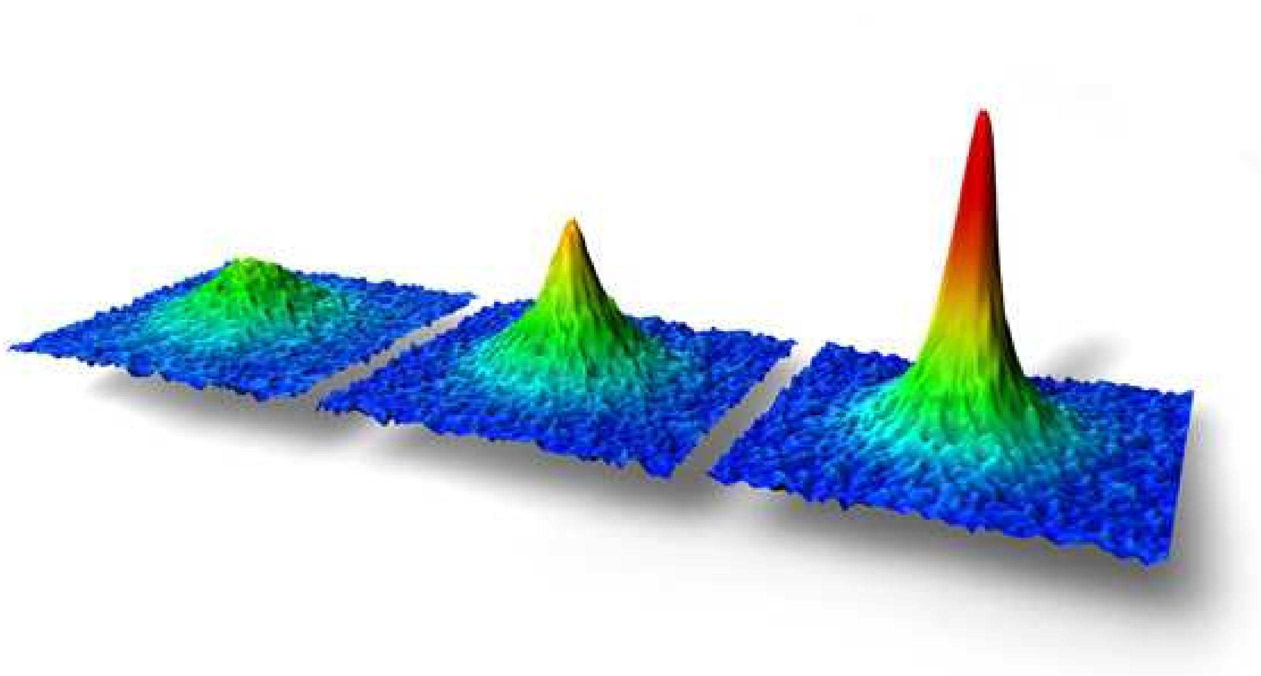
Resonantly enhanced tunnelling

Wannier-Stark ladder of energy levels: $E_m(\alpha) = E_0(\alpha) + F d_L m$

resonant tunnelling for $F \approx \dfrac{E_1 - E_0}{(2\pi m)}$ (with integer m)



2. Bose-Einstein condensate



3D Gross-Pitaevskii equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) = \left[-\frac{\hbar^2}{2M}\nabla^2 + \frac{1}{2}M\left(\omega_x^2 x^2 + \omega_\mathrm{r}^2\rho^2\right) + V\sin^2\left(\frac{\pi x}{d_L}\right) + Fx + g\left|\psi(\vec{r},t)\right|^2\right]\psi(\vec{r},t)$$

with the characteristic scales defined by the lattice period d_L:

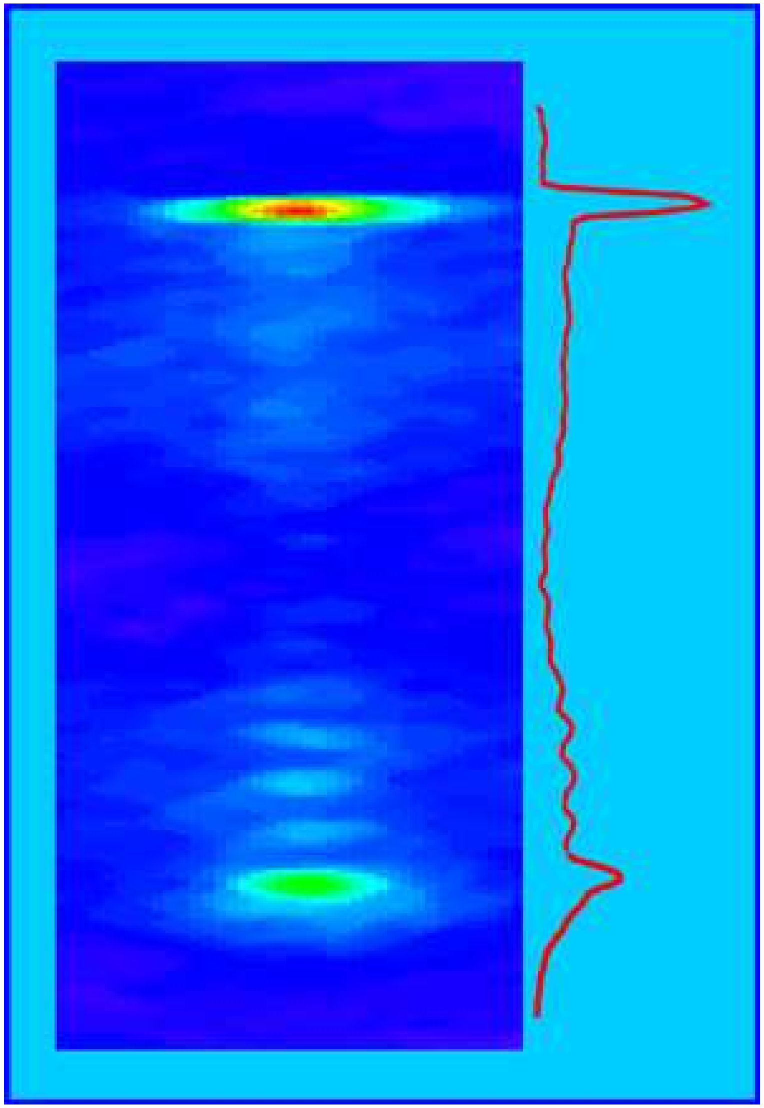
$E_\mathrm{R} = p_\mathrm{R}^2/2M$ for $p_\mathrm{R} = \hbar\pi/d_L$

and the dimensionless parameters:

$$x = x_\mathrm{SI}2\pi/d_L, V_0 = V_\mathrm{SI}/8E_\mathrm{R}, F = F_\mathrm{SI}d_L/(16\pi E_R), g = g_\mathrm{SI}d_L N/(16\pi E_\mathrm{R})$$

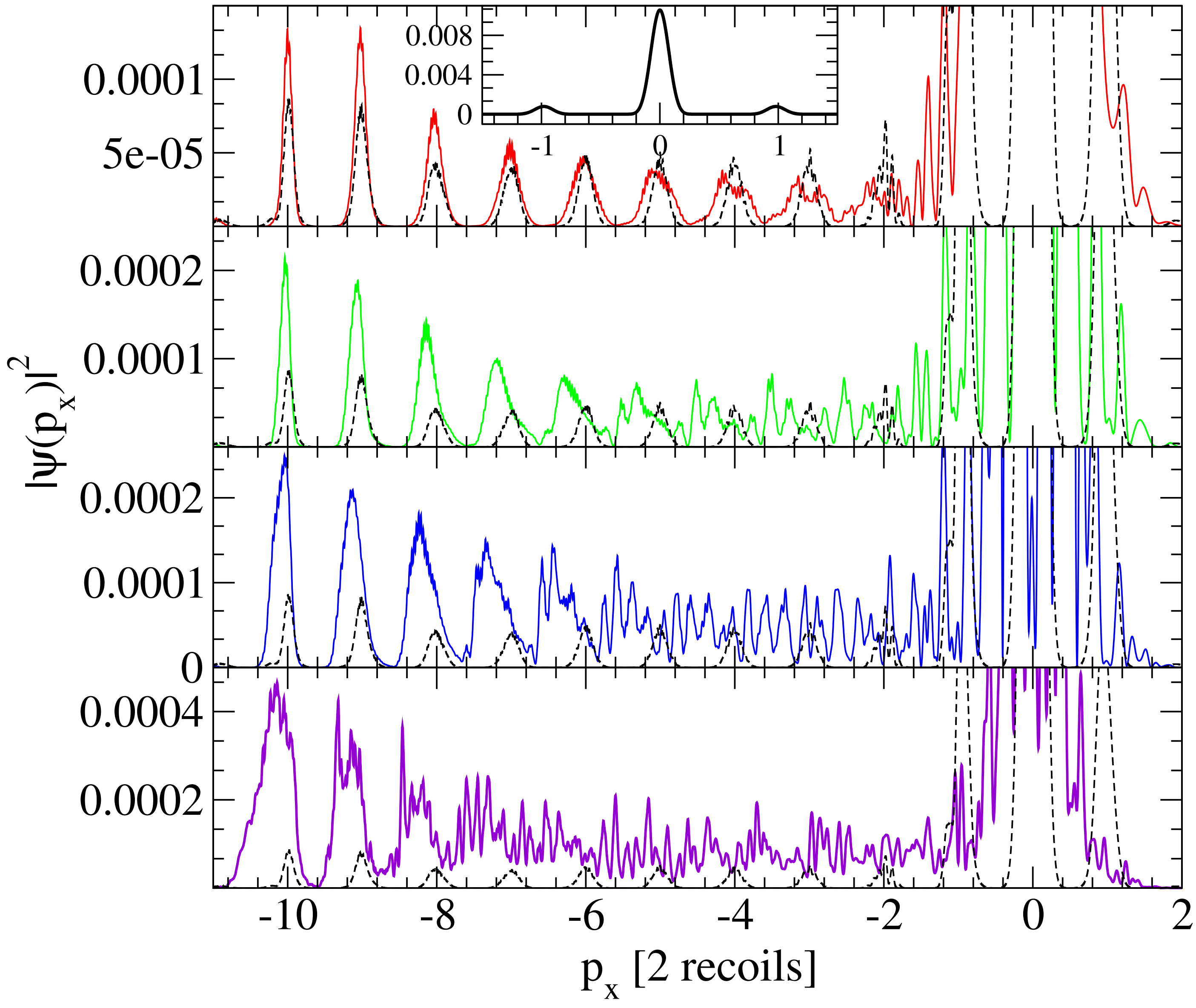
Experimental observable: Momentum distribution

Experiment:



Experimental observable: Momentum distribution

3D GPE:



Survival and recurrence probabilities

Survival probability:

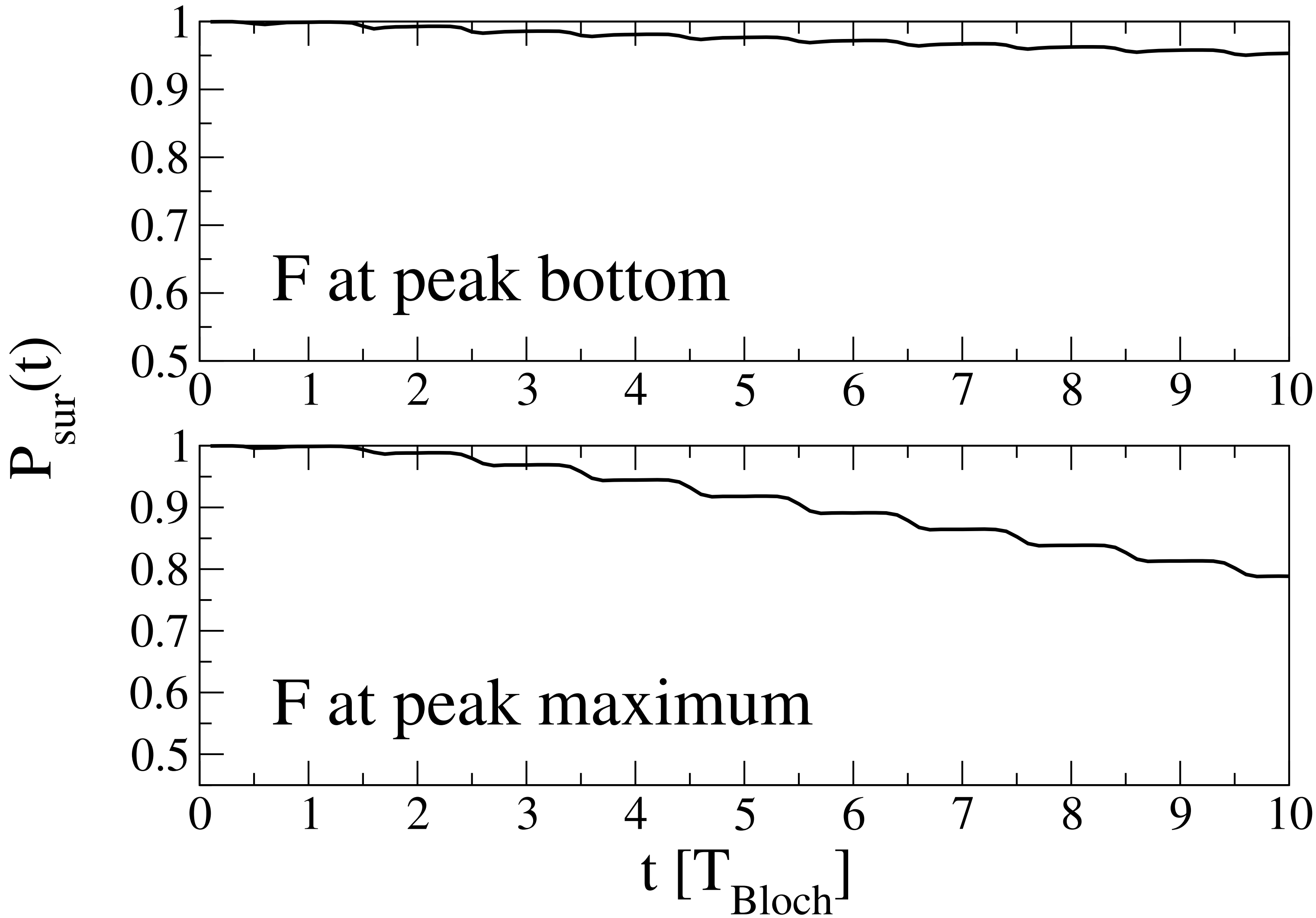
$$P_{\mathrm{sur}}(t) \equiv \int_{-p_{\mathrm{c}}}^{p_{\mathrm{c}}} dp_x \left(\int dp_y dp_z |\psi(\vec{p}, t)|^2 \right) \ ,$$

where $p_{\mathrm{c}} \geq 3p_{\mathrm{R}}$

Recurrence probability = autocorrelation:

$$P_{\mathrm{rec}}(t) \equiv |\langle \psi(t) | \, \psi(t=0) \rangle|^2 \ .$$

Survival probability and tunnelling rates:



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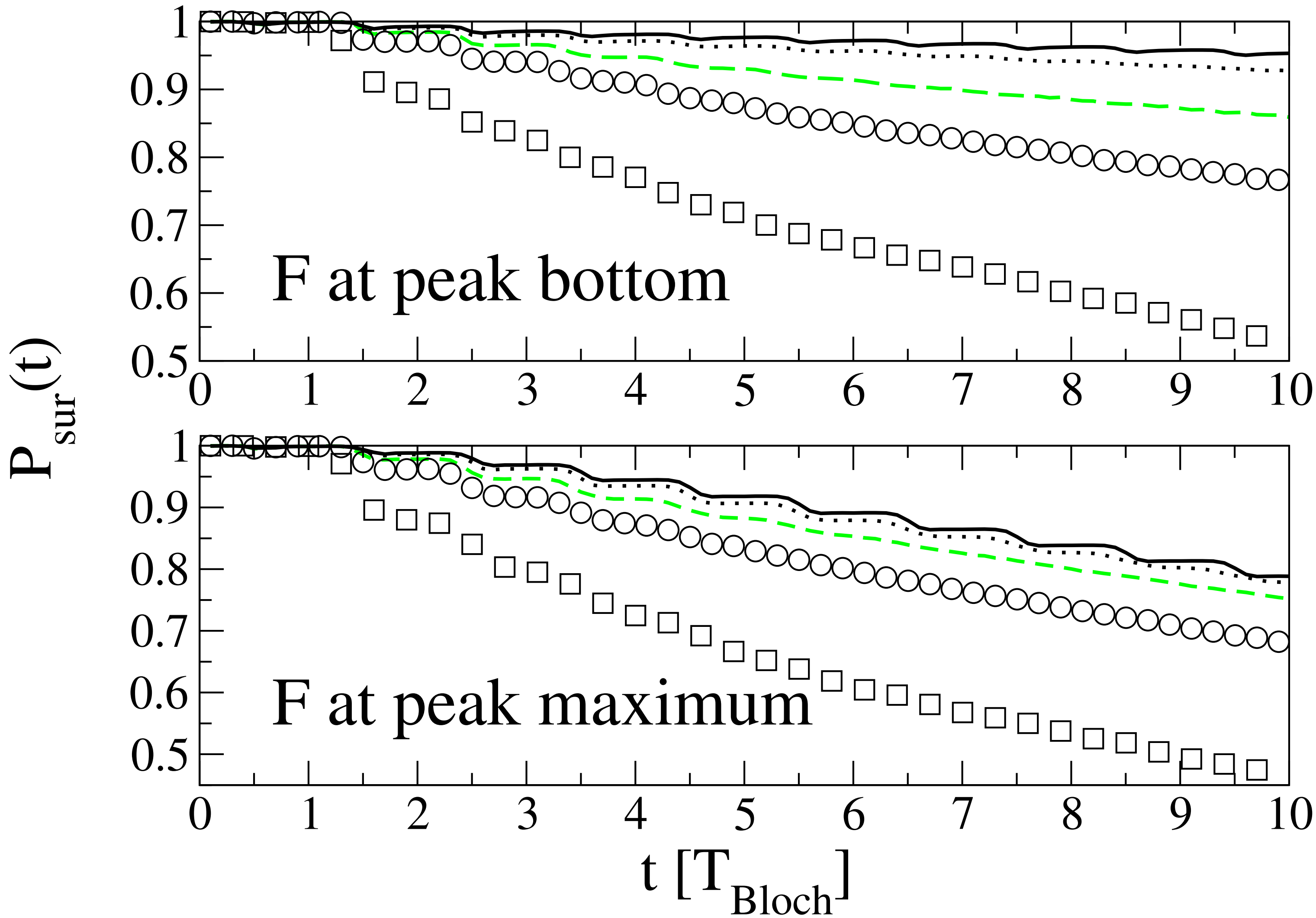

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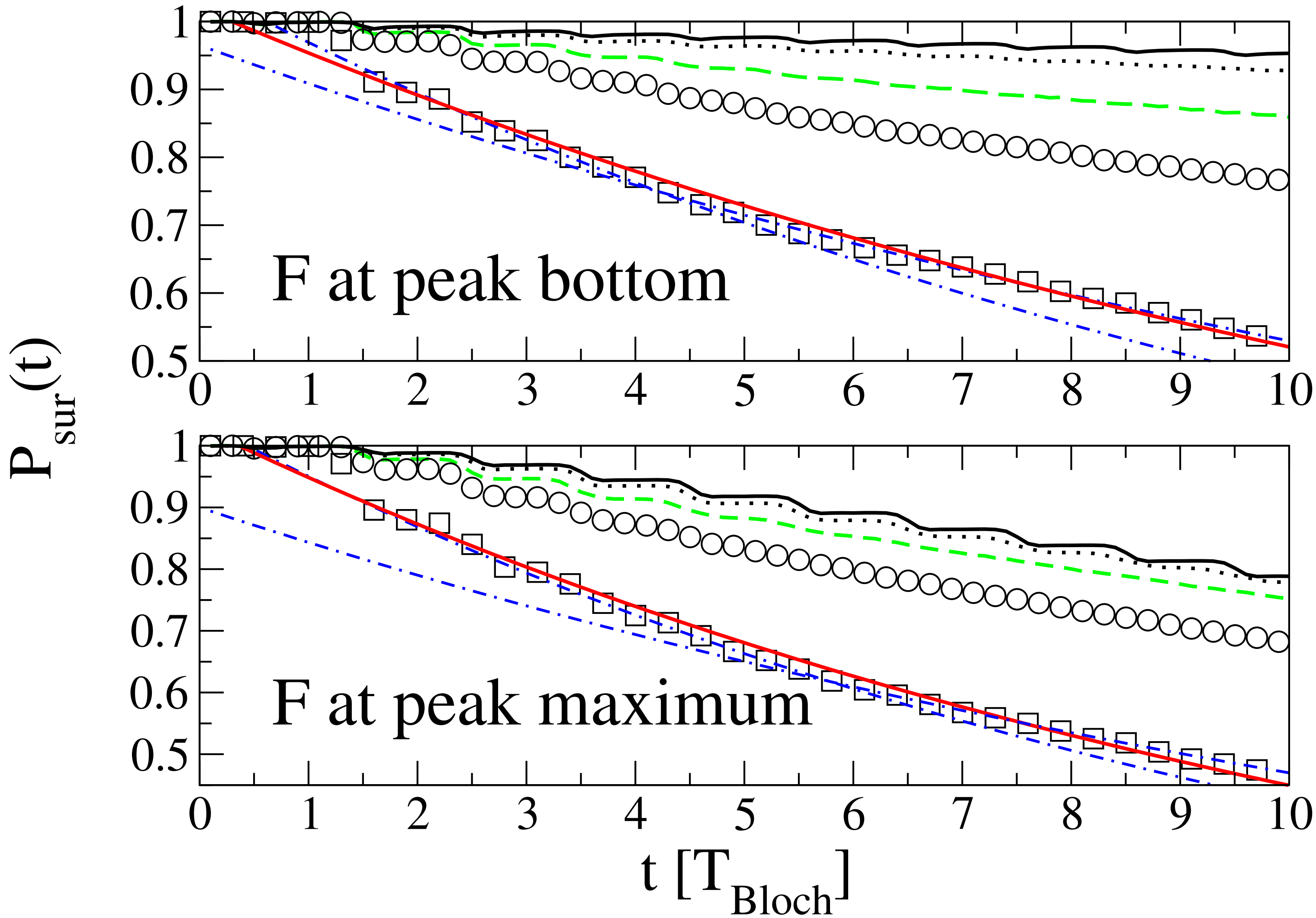

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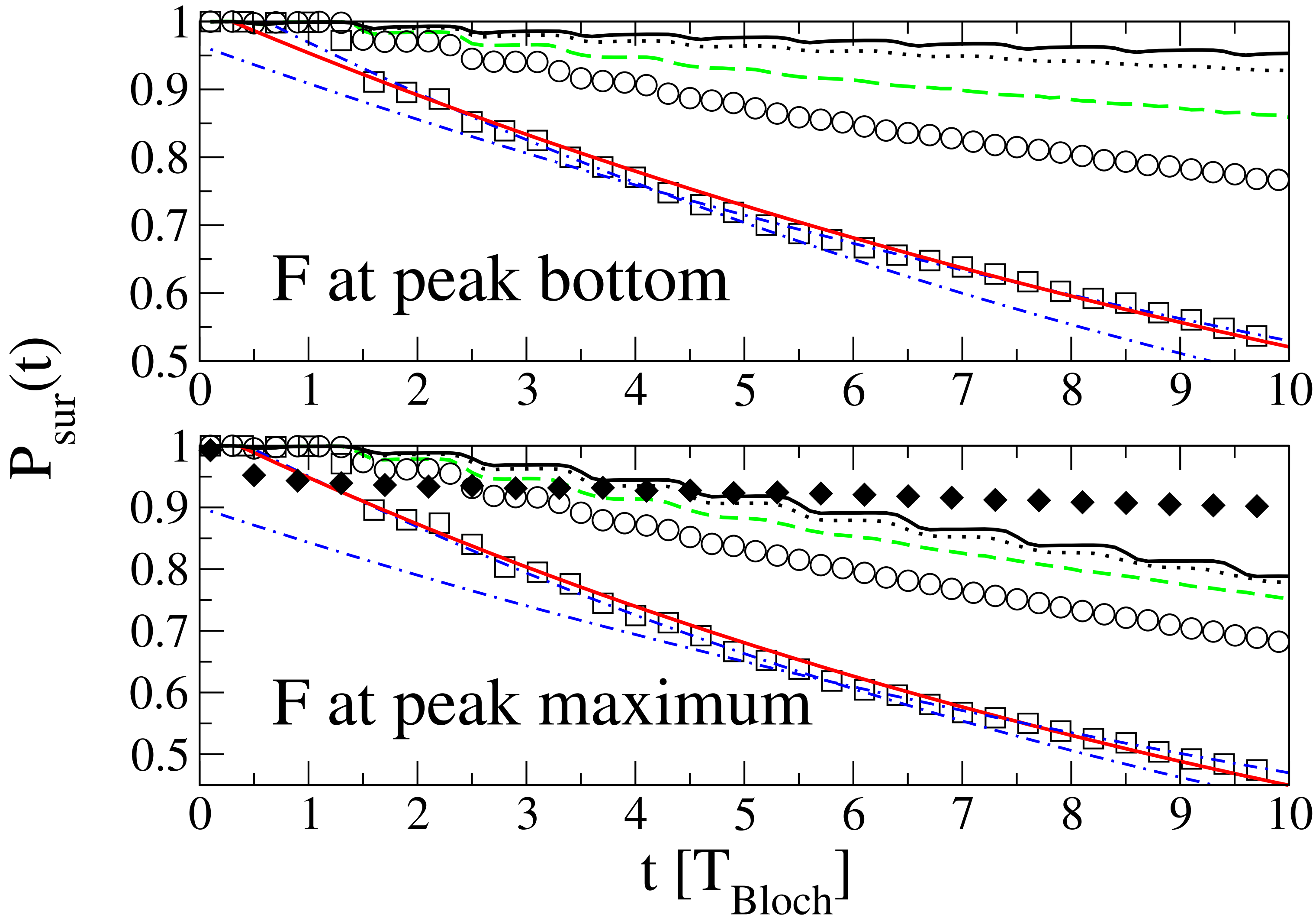

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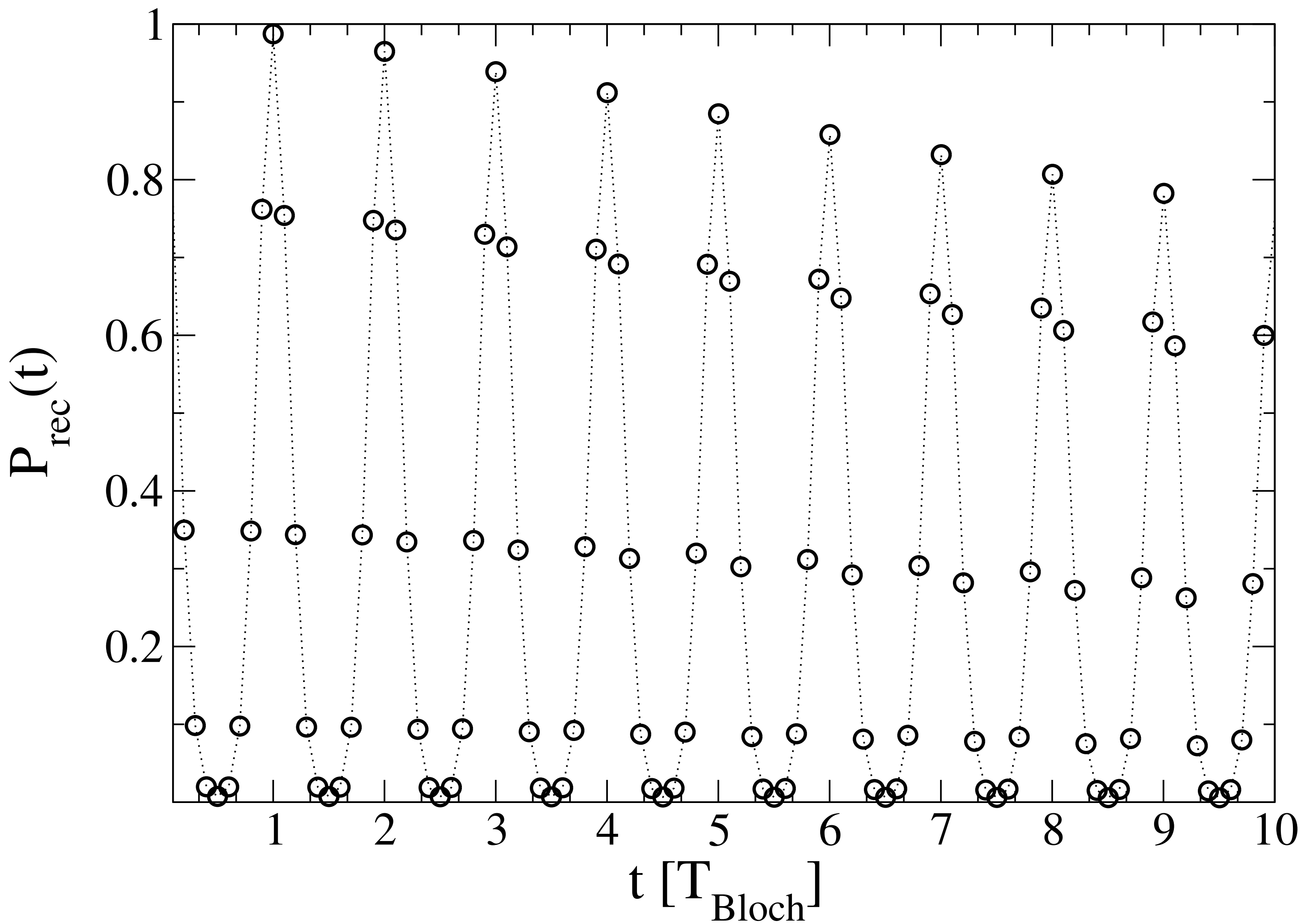

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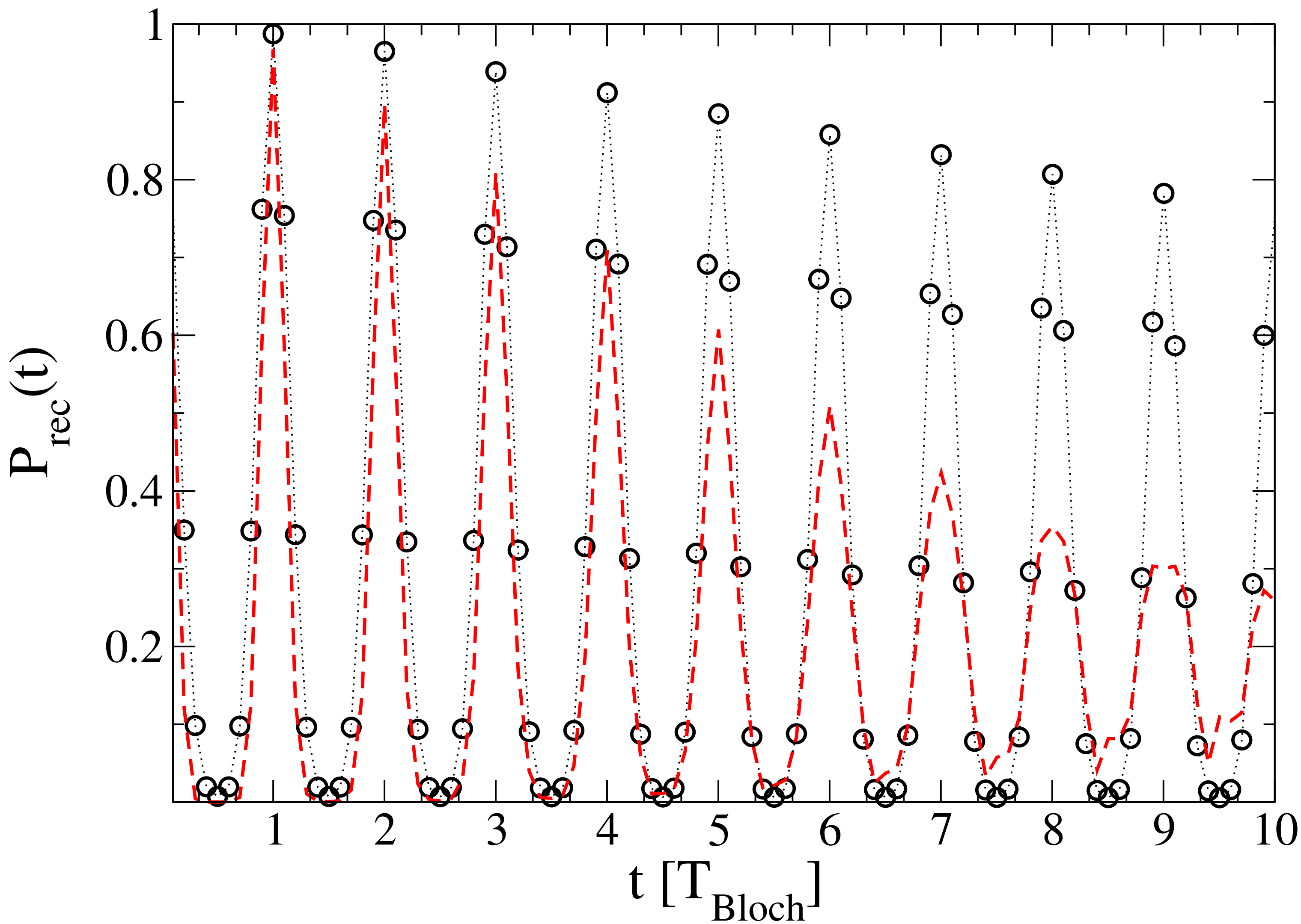

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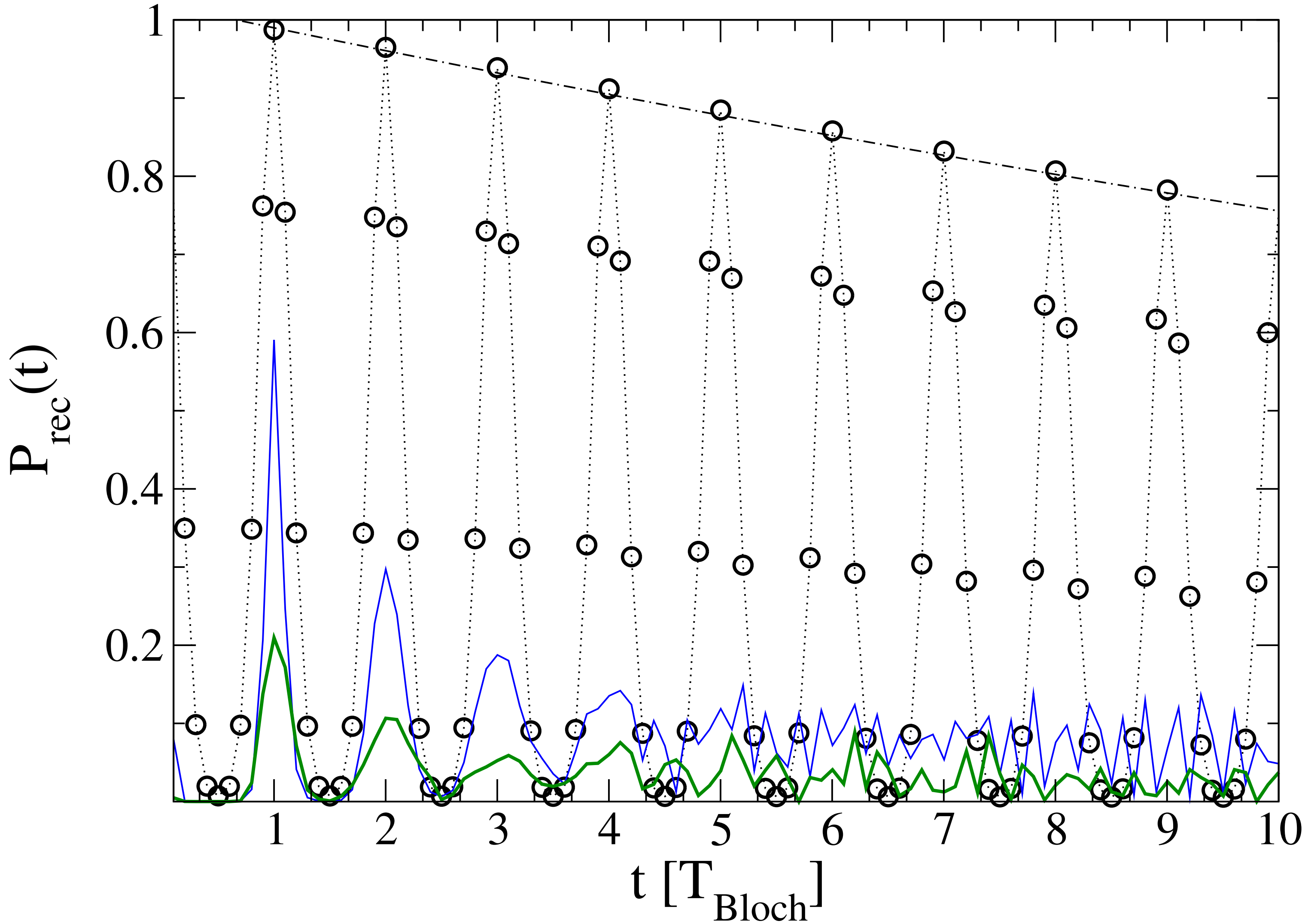
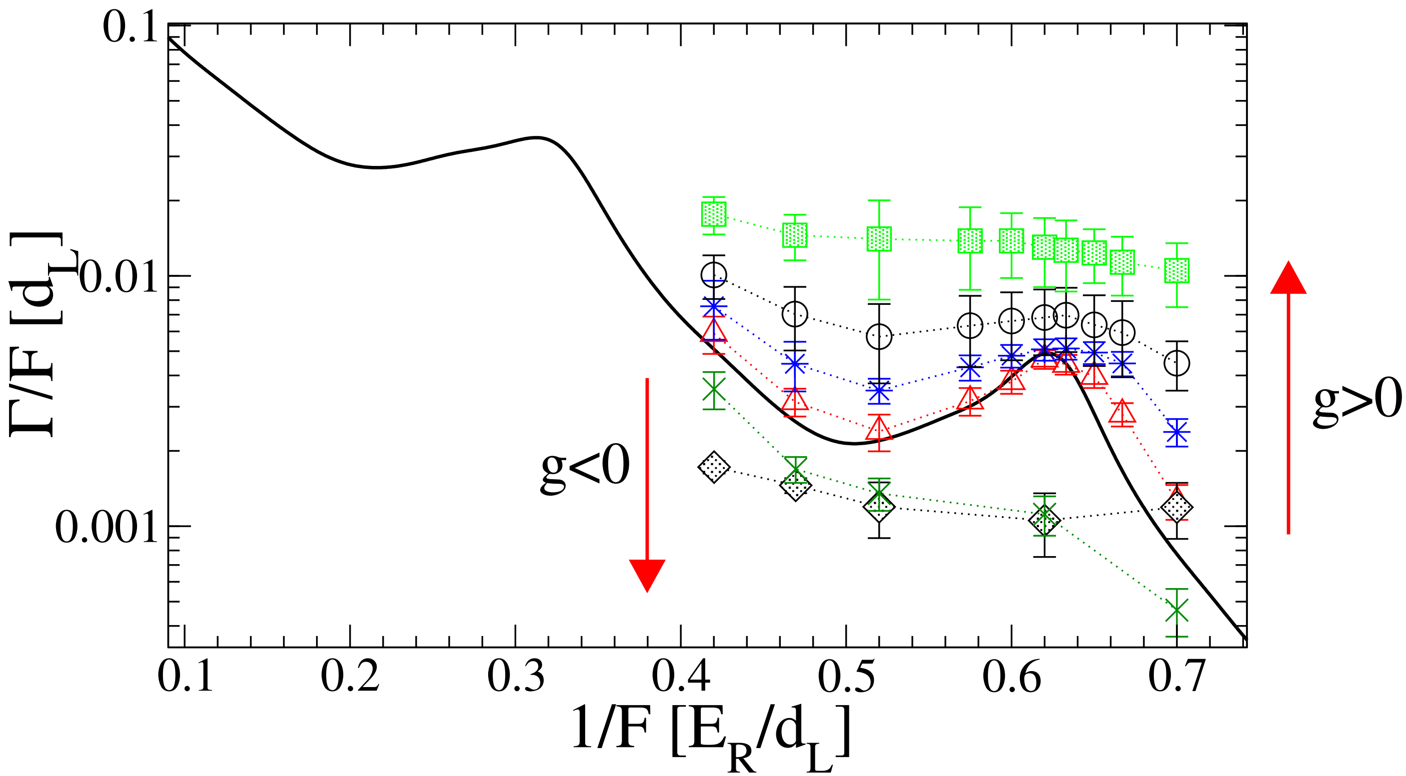


3. Nonlinear tunnelling rates?

For $g = 0$ we can define characteristic rates Γ:

$$P_{\rm sur}(t) \sim \exp\left(-\frac{t\Gamma}{\hbar}\right)$$



Nonlinear resonance states?

Since $V(x) = V_0 \sin^2(x) + Fx \to -\infty$ for $x \to -\infty$, any wave packet initially prepared in the optical lattice will escape!!

resonance state ψ_g:

$$H[\psi_g]\psi_g = E_g \psi_g \ ,$$

for $E_g = \mu_g - i\Gamma_g/2$ and

$$H[\psi] = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + V(x) + g|\psi(x)|^2$$

To render the problem meaningful we demand that the condensate wave function remains normalised in the central potential well around $x \approx 0$:

$$\int_{-x_c}^{x_c} dx \, |\psi_g(x)|^2 = 1 \ .$$

Complex-scaling technique

$$\psi(x) \to \psi^\theta(x) \equiv \hat{R}(\theta)\psi(x) \equiv e^{\,\mathrm{i}\theta/2}\psi(xe^{\,\mathrm{i}\theta}) \ ,$$

$$\overline{\psi(x)} \to \overline{\psi}^\theta(x) \equiv \hat{R}(\theta)\overline{\psi}(x) \equiv e^{\,\mathrm{i}\theta/2}\overline{\psi}(xe^{\,\mathrm{i}\theta})$$

$$\Rightarrow \overline{\psi}^\theta(x) = \hat{R}(\theta)\left(\hat{R}(-\theta)\psi^\theta\right)^*(x)$$

Eigenvalue problem in the complex domain:

$$H^\theta[\psi_g^\theta]\psi_g^\theta = E_g\psi_g^\theta \ ,$$

with

$$H^\theta[\psi_g^\theta] = -\frac{1}{2}\frac{\partial^2}{\partial x^2}e^{-\,\mathrm{i}2\theta} + V(xe^{\,\mathrm{i}\theta}) + g_\theta\overline{\psi}_g^\theta(x)\psi_g^\theta(x)$$

and

$$g_\theta = ge^{-\,\mathrm{i}\theta}$$

Numerical implementation

Starting with an initial guess for $\psi^\theta(x, t = 0)$, we evolved in (real-)time the grid representation of $\psi^\theta(x, t)$:

$$\psi^\theta(x,t) = \sum_{j=-n}^{n} c_j(t)\chi_j(x) ,$$

with the box functions

$$\chi_j(x) = \begin{cases} 1/\Delta_x & , \quad |x/\Delta_x - j| < 1/2 \\ 0 & , \quad \text{otherwise} \end{cases}$$

and a suitable grid spacing Δ_x.

Numerical algorithm

"forward time centred space" representation of the derivatives leads to a tridiagonal Hamiltonian matrix which is evolved by

Crank-Nicholson steps:

$$\left(1 + \mathsf{i}\, H^\theta[\psi^\theta]\Delta t/2\right)\psi^\theta(x, t + \Delta t/2) = \left(1 - \mathsf{i}\, H^\theta[\psi^\theta]\Delta t/2\right)\psi^\theta(x, t - \Delta t/2)\ .$$

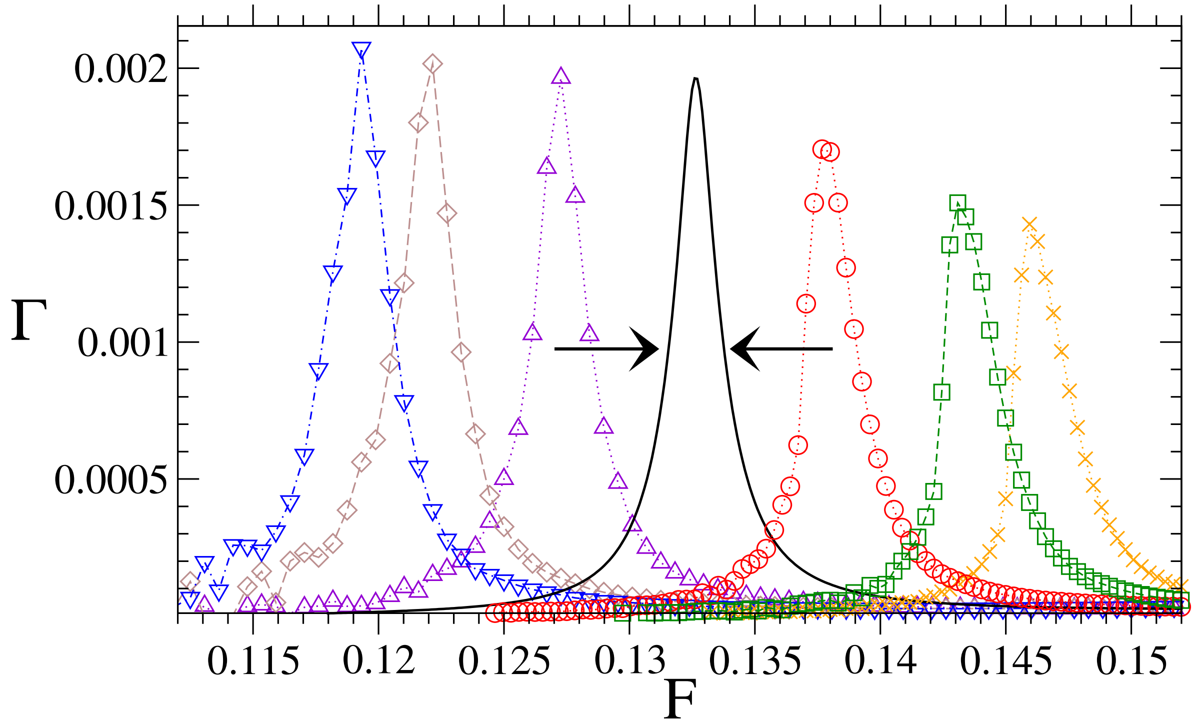
and then iterated in the following way:

$$\psi^\theta(x, t + \Delta t) = \left(1 - \mathsf{i}\, H^\theta[\psi^\theta]\Delta t\right)\psi^\theta(x, t)$$

to arrive at the resonance state for sufficiently large t:

$$\psi^\theta(x, t) \to \psi_g^\theta(x)$$

Result for nonlinear tunnelling rates



$$\Delta E \approx g \int_{-x_c}^{x_c} dx \, |\psi_g|^2 \, |\psi_{g=0}|^2 \approx g \int_{-\pi}^{\pi} dx \, |\psi_{g=0}|^4$$

$$\Rightarrow \frac{2\pi \Delta F}{|g|} \approx \int_{-\pi}^{\pi} dx \, |\psi_{g=0}|^4 \approx 0.37$$

4. Conclusions

* quantum transport models realisable with Bose-Einstein condensates in optical lattices: Nonlinear Wannier-Stark problem

* Robust experimental observables for tunnelling decay: survival and recurrence probabilities

* Nonlinear resonance states $\Leftrightarrow$ renormalisation $\&$ complex-scaling algorithm

* Interaction-induced shift and broadening of RET peaks

Open problem:

Computation of survival probability (remaining atoms $N(t)$):

$$\frac{dN(t)}{dt} = -\Gamma_{g(t)} N(t) \ ?$$

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Riccardo Mannella and Ennio Arimondo (Unversità degli Studi di Pisa)

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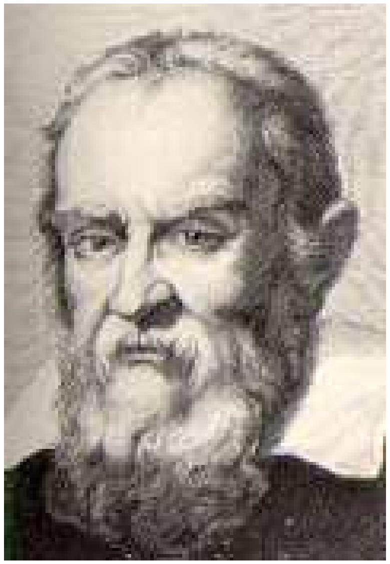

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