Anomalous Lévy diffusion:
From the flight of an albatross to optical lattices

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Outline

1. Lévy distributions
   - Broad distributions
   - Central–limit theorem
   - Physical examples
2. Lévy diffusion
   - Normal diffusion
   - Anomalous diffusion (*fractional derivative)*
   - Experimental observations
3. Statistical mechanics
   - Anomalous Lévy diffusion in an optical lattice
   - Fluctuations: how large is large?
   - Ergodicity

Broad distributions

Physical point of view

- Many phenomena: well-defined average behavior with fluctuations around average value

  $\longrightarrow$ fluctuations described by narrow distributions

  ex: Gauss distribution

- Other phenomena: properties dominated by fluctuations (average not important, may not even exist)

  $\longrightarrow$ fluctuations described by broad distributions

  ex: Lévy distributions

"broad"=power-law tail

Broad distributions

Mathematical point of view: Central–limit theorem

$$S_N = X_1 + X_2 + ... + X_N$$

Sum of N *independent* random variables X_i

- Case 1: finite second moment $\langle X_i^2 \rangle < \infty$

  $S_N \to$ Gauss distribution $\qquad N \to \infty$

- Case 2: divergent second moment $\langle X_i^2 \rangle \to \infty$

  $S_N \to$ Lévy distribution $\qquad N \to \infty$

Stable probability distributions

Lévy distribution

Definition and main properties: Lévy 1937

- Characteristic function

$$\varphi(k) = \int dx\, P(x)\, e^{-ikx} = e^{-c|k|^{\alpha}} \qquad 0 < \alpha \leq 2$$

- Asymptotic power–law decay

$$P(x) \sim \frac{1}{|x|^{1+\alpha}}, \qquad x \to \infty \qquad \alpha < 2$$

- Second moment diverges

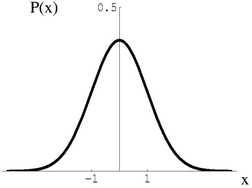
$$\langle x^2 \rangle = \int dx\, x^2\, P(x) \to \infty \qquad \alpha < 2$$

Scale-free distributions

Physical examples

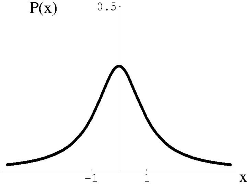
$\alpha = 2$ Gauss $\frac{e^{-x^2/2}}{\sqrt{2\pi}}$

(Errors in astronomical observations) 1809



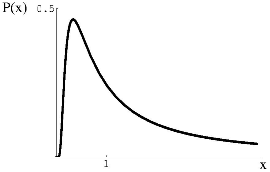
$\alpha = 1$ Breit-Wigner $\frac{1}{\pi}\frac{1}{1+x^2}$

(Cross-section of resonant scattering) 1936



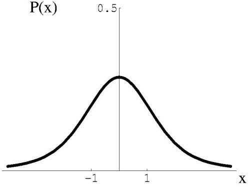
$\alpha = \frac{1}{2}$ Schrödinger $\frac{e^{-1/2x}}{\sqrt{2\pi x^3}}$

(First-passage time distribution) 1915



$\alpha = \frac{3}{2}$ Holtzmark

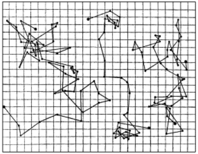
(Random gravitational force on a star) 1919

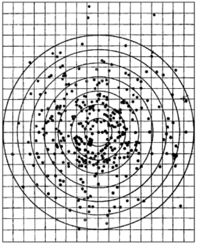


Outline

Normal diffusion

Example: small particles in a glass of water Perrin 1909





Irregular, random motion

$\longrightarrow$ Brownian motion

Brown 1827

- Gaussian dynamics:

  Diffusion equation:

$$\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial x^2}$$

  Solution:

$$P(x,t) = \frac{1}{\sqrt{4\pi D t}} e^{-x^2/(4Dt)}$$

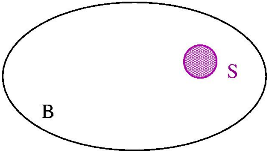
- Mean-square displacement:

$$\langle x^2 \rangle = 2Dt$$

$\longrightarrow$ linear in time

Transport equations: Gauss

System S in contact with bath B
(temperature T)



Langevin 1908

$$m\ddot{x} + U'(x) + \eta\dot{x} = F(t)$$

$\eta =$ friction

$F(t) =$ Gaussian random force (central-limit theorem)

$\langle F(t)\rangle = 0$

$\langle F(t)F(t')\rangle = 2D\,\delta(t - t')$ $\quad D = \eta kT$

Lévy diffusion

- Definition:

$$m\ddot{x} + U'(x) + \eta\dot{x} = F(t)$$

$F(t) =$ Lévy random force

(central-limit theorem)

- Properties:

algebraic tail of force distribution

$\longrightarrow$ large fluctuations

$\longrightarrow$ long jumps (Lévy flights)

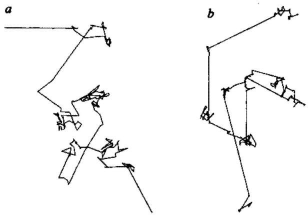
Enhanced diffusion ($\langle x^2 \rangle \to \infty$)

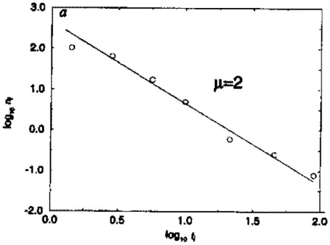
Flight of an albatross

Flight-time measurement of 30 albatrosses at Bird Island

Typical trajectories:

Stanley et al., Nature 1996





Distribution of flight-time intervals:

$$P(t_i) \sim \frac{1}{(t_i+1)^{\mu}} \qquad \mu = 1+\alpha \simeq 2$$

Power-law distribution

Experiment 1: Diffusion in micelles

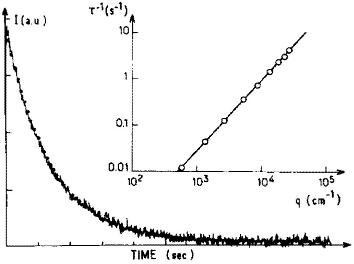
Ott et al., PRL 1991

Micelles = polymer-like molecules that break and recombine rapidly

Measurement: intensity of light emitted by fluorescent atoms after photobleaching

$\longrightarrow$ characteristic function

$\alpha = 1.5$



Diffusion equation:

Gauss $\quad \frac{\partial P}{\partial t} = D\,\frac{\partial^2 P}{\partial x^2} \longrightarrow \frac{\partial \widetilde{P}}{\partial t} = -Dk^2\,\widetilde{P} \longrightarrow \widetilde{P}(k,t) = e^{-Dt\,k^2}$

Lévy $\quad \frac{\partial P}{\partial t} = D\,\frac{\partial^\alpha P}{\partial |x|^\alpha} \longleftarrow \frac{\partial \widetilde{P}}{\partial t} = -D|k|^\alpha \widetilde{P} \longleftarrow \widetilde{P}(k,t) = e^{-Dt\,|k|^\alpha}$

Fractional derivative

- Letter from Leibniz to L'Hospital: $\frac{d^{1/2}}{dx^{1/2}} x = ?$ 30.09.1695

$$\frac{d^m}{dx^m} x^n = (n-m)!\, x^{n-m} = \Gamma(n-m+1)\, x^{n-m}$$

$$m = 1/2, n = 1 \longrightarrow \frac{d^{1/2}}{dx^{1/2}} x = \Gamma(3/2)\, x^{1/2}$$

- Riemann-Liouville operator

$$\frac{d^\alpha}{dx^\alpha} F(x) \longleftrightarrow (ik)^\alpha \widetilde{F}(k) \qquad \text{Ex: } \frac{d^\alpha}{dx^\alpha} e^{ax} = a^\alpha\, e^{ax}$$

- Riesz operator

$$\frac{d^\alpha}{d|x|^\alpha} F(x) \longleftrightarrow -|k|^\alpha \widetilde{F}(k) \qquad \text{Ex: } \frac{d^\alpha}{dx^\alpha} \cos ax = -|a|^\alpha \cos ax$$

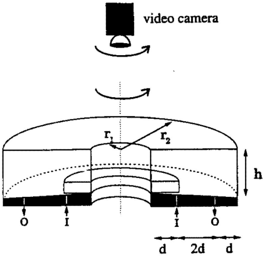
$$\frac{d^\alpha}{d|x|^\alpha} F(x) = \frac{1}{\kappa_\alpha} \frac{d^2}{dx^2} \int dy\, \frac{F(y)}{|x-y|^{\alpha-1}} \longrightarrow \text{non-local}$$

Experiment 2: 2D rotating flow

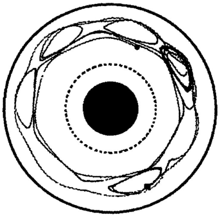
Weeks et al., PRL 1993

Rotating tank (water + glycerol)

1.5 Hz

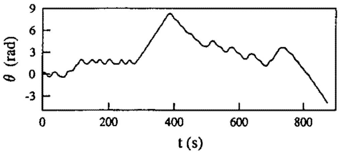


Typical trajectory:

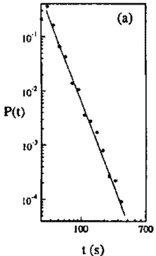


Regime of chaotic advection

⟶ chain of vortices



Pdf of duration of flight:



$\alpha = 1.3$

Experiment 3: Surface diffusion

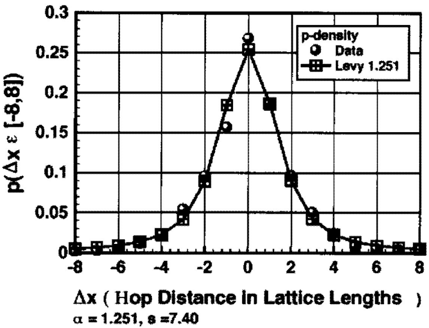
Senft et al., PRL 1995

Palladium atoms on a tungsten lattice $T = 133\text{K}$

$\alpha = 1.251$, $s = 7.4$

- Measurement:
  displacement of *single* atom on a 1D lattice
  (field microscopy)

- Low diffusion barrier
  $\longrightarrow$ longs jumps
  (2, 3 or more lattice sites)

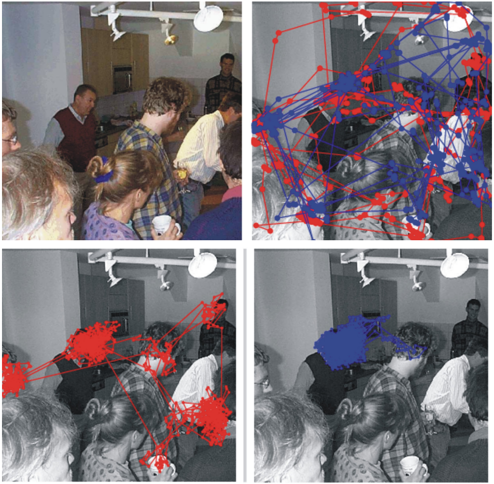


$$\varphi(k) = e^{-|sk|^{\alpha}}$$

Realization of 1D discrete random walk

Experiment 4: Saccadic eye movements

Brockmann and Geisel 2004

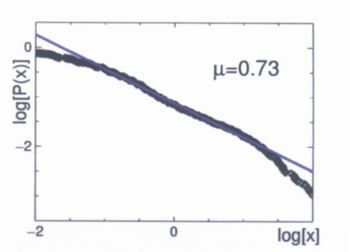


- Magnitude of visual angle: $0.1^\circ < x < 100^\circ$

  $P_{exp}(x > 75^\circ) = 3 \times 10^{-7}$

- Pdf of saccadic magnitude larger than x:

  $$P_>(x) = \int_x^\infty dx' \, x'^{-(1+\alpha)}$$
  $$\sim x^{-\alpha} \qquad (\alpha = \mu)$$



Outline

Optical lattice

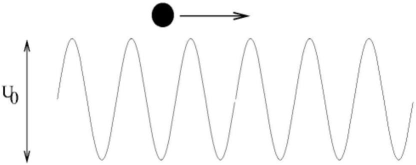
- Spatially periodic optical potential:

  obtained, for instance, by superposition of two counterpropagating laser beams

  $\equiv$ "optical crystal" (band structure, Bragg scattering, ...)

- Advantages:

  $\longrightarrow$ potential is exactly known (no defects)
  $\longrightarrow$ can be tuned in a precise way (spacing, amplitude)



Atomic transport?

Atomic transport in optical lattice

- Transport equation for the semiclassical Wigner function:

Dalibard and Cohen-Tannoudji 1989

$$\frac{\partial W(p,t)}{\partial t} = -\frac{\partial}{\partial p}[K(p)W] + \frac{\partial}{\partial p}[D(p)\frac{\partial W}{\partial p}]$$

Drift : $K(p) = -\frac{\bar{\alpha} p}{1+(p/p_c)^2}$ $\longrightarrow$ Cooling

[Friction force] (Sisyphus effect)

Diffusion : $D(p) = D_0 + \frac{D_1}{1+(p/p_c)^2}$ $\longrightarrow$ Heating

[Momentum fluctuations]

- Range of validity:
  - large momentum, $p \gg \hbar k$ — defines semiclassical limit
  - small saturation, $s \ll 1$ — low laser intensity
  - large kinetic energy, $p^2/2m \gg U_0$ — allows spatial averaging

Power-law tail distribution in momentum

Stationary momentum distribution:

$$W_s(p) = \frac{1}{Z}\left[1 + \frac{D_0}{D_0 + D_1}\frac{p^2}{p_c^2}\right]^{-(\bar{\alpha}p_c^2)/(2D_0)}$$

Power-law tail: $W_s(p) \sim |p|^{-(\bar{\alpha}p_c^2)/D_0}$

Exponent can be rewritten as:

$$(\bar{\alpha}p_c^2)/D_0 = U_0/(22E_R)$$

(U_0 = potential depth, E_R = recoil energy)

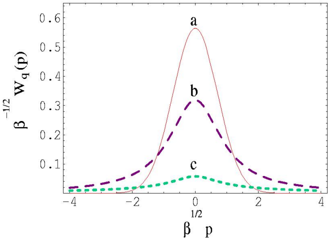
$\longrightarrow$ Statistical properties of $W_s(p)$ can be changed:

– from normal statistics for $U_0 > 66E_R$

– to Lévy statistics for $U_0 < 66E_R$

In fact, $W_s(p) =$ *Tsallis distribution* Lutz PRA (R) 2003

Lévy statistics and diverging moments



- Second moment: $\langle p^2 \rangle = \int dp\, p^2\, W_s(p)$

  diverges if $U_0 < 66 E_R$.

  $\longrightarrow$ mean kinetic energy, $\langle E_K \rangle = \langle p^2 \rangle / 2m$, infinite.

- Fourth moment: $\langle p^4 \rangle = \int dp\, p^4\, W_s(p)$

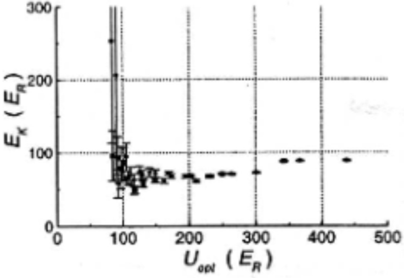
  diverges if $U_0 < 110 E_R$.

  $\longrightarrow$ mean square kinetic energy, $\langle E_K^2 \rangle = \langle p^4 \rangle / 4m^2$, infinite.

Experiment

Single $^{24}Mg^{+}$ ions in a one–dimensional optical lattice:

Katori et al., PRL 1997



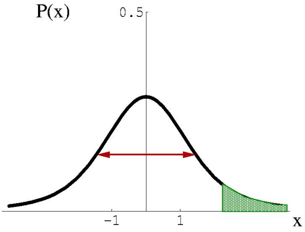
Experimental observation of the divergence of the atomic kinetic energy

Fluctuations: how large is large?

Statistical properties of rare but extreme events:

- How tall should one design an embankment so that the river reaches this level only once in 50 years?
- How large will be the largest earthquake in Los Angeles in the next 20 years?
- How large is the largest atomic momentum in 1000 observations?

Extreme Value Theory   Gumbel 1958



Fluctuations: how large is large?

For independent events:

Value $x_{max} = \lambda$ of the maximum among N realizations that will not be exceeded with probability p satisfies:

$$\int_{\lambda}^{\infty} dx\, P(x) \simeq \frac{\ln(1/p)}{N}$$

- for exponential $P(x) = e^{-x}$ $\qquad \lambda \sim \ln\left[\frac{N}{\ln(1/p)}\right]$
- for power-law $P(x) = x^{-(1+\alpha)}$ $\qquad \lambda \sim \left[\frac{N}{\ln(1/p)}\right]^{1/\alpha}$

Example: $P(x) = C\,(1 + x^2)^{-(1+\alpha)/2}$ $\quad N = 1000 \quad p = 99\%$

U_0/E_R	α	λ		
440	19	2		
110	4	27	1.3×10	($\langle p^4 \rangle$ diverges)
66	2	631	3×10^2	($\langle p^2 \rangle$ diverges)
44	1	313 000	10^5	($\langle p \rangle$ diverges)

Ergodicity

Motivation: Two recent experiments

- "Experimental Investigation of Nonergodic Effects in Subrecoil Laser Cooling" Saubamea et al., PRL 1999
- "Statistical Aging and Nonergodicity in the Fluorescence of Single Nanocrystals" Stockmann et al., PRL 2003

$\longrightarrow$ Lévy statistics induces ergodicity breaking

Question: connection between nonergodicity and diverging moments?

Advantage of optical lattice:

$\longrightarrow$ Ordinary linear Fokker-Planck equation

Ergodicity

Definition:

"Ensemble average and time average of observables are equal in the infinite-time limit."

$$\langle A \rangle = \overline{A} \qquad \text{as} \qquad t \to \infty$$

- Ensemble average: $\langle A \rangle = \int dp\, A(p)\, W(p,t)$

  in the long–time limit: $\langle A \rangle_s = \int dp\, A(p)\, W_s(p)$

- Time average : $\overline{A} = \frac{1}{t} \int_0^t d\tau\, A(p(\tau))$

Ergodicity in the mean-square sense:

$$\sigma_A^2(t) = \langle \left(\overline{A} - \langle \overline{A} \rangle \right)^2 \rangle \to 0 \quad \text{when} \quad t \to \infty$$

Ergodicity

Ergodicity breaking: Lutz PRL 2004

Ergodicity depends in general on the observable $A(p)$

for $A(p) = p^n$ $\quad \sigma_A^2(t) \sim t^{-\mu}$ $\quad$ with $\quad \mu = \dfrac{1-(2n+1)D}{2D}$

$\longrightarrow$ $\sigma_A^2(t) \to 0$ as $t \to \infty$ $\quad$ only if $\quad D < D_n = 1/(2n+1)$

Existence of a noise threshold D_n above which ergodicity is broken

(rescaled variables $\bar{\alpha} = p_c = 1, D_1 = 0, D_0 = D = 22E_R/U_0$)

Ergodicity

- Moments of stationary momentum distribution:

$$\langle p^m \rangle = \int dp\, p^m\, W_s(p)$$

are finite for $D < D'_m = 1/(m+1)$

Direct relationship between the loss of ergodicity for $A(p) = p^n$ and the divergence of the $2n$th moment of the stationary momentum distribution

In particular:

for $n = 1$ $\quad \langle p \rangle \neq \overline{p}$ when $\langle p^2 \rangle$ diverges $(D = 1/3, U_0 = 66 E_R)$

Calculation of $\sigma_A^2(t)$

$$\sigma_A^2(t) = \frac{1}{t^2}\int_0^t dt_1 \int_0^t dt_2 \Big[[\langle A(p(t_1))A(p(t_2))\rangle - \langle A(p(t_1))\rangle \langle A(p(t_2))\rangle \Big]$$

Two-time correlation function:

$$\langle A(p(t_1))A(p(t_2))\rangle = \int\int dp_1 dp_2\, A(p_1)A(p_2)\, W_2(p_1,t_1;p_2,t_2)$$

Two-point joint probability density function:

$$W_2(p_1,t_1;p_2,t_2) = \psi_0(p_1)\psi_0(p_2)\Big(\psi_0(p_1)\psi_0(p_2)$$

$$+\int_0^\infty dk\;\psi_k(p_1)\psi_k(p_2)\,e^{-E_k|t_1-t_2|}\Big)$$

Finally $\sigma_A^2(t) = \frac{2}{t^2}\int_0^t d\tau\,(t-\tau)\,C_A(\tau)$

With $C_A(\tau) = \int_0^\infty dk\,\left[\int dp\,A(p)\,\psi_0(p)\psi_k(p)\right]^2 e^{-E_k\tau}$

Power-law tail distribution in time

- First passage time distribution:

  Divide momentum space in two regions:

  – low momentum region $p < 1$, $\quad K_1(p) \sim -p$

  – high momentum region $p > 1$, $\quad K_2(p) \sim -1/p$

First–passage time = time at which the momentum of the system first exits a certain momentum interval, given that it was originally in that interval

In the high momentum region:

$$g_2(t) \sim t^{-\gamma} \qquad \text{with} \qquad \gamma = (3D+1)/(2D)$$

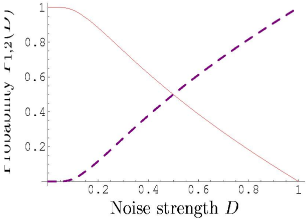
Power-law distribution in time

Power-law tail distribution in time

- Moments of the first passage time distribution:

$$\langle t^n \rangle = \int dt\, t^n\, g_2(t)$$

converge for $D < D_n'' = 1/(2n-1)$



Ergodicity is broken for $A(p) = p^n$ when the $(n+1)$th moment of the first passage time distribution in the high–momentum region becomes infinite

In particular:

for $n = 1$ $\langle p \rangle \neq \overline{p}$ when $\langle t^2 \rangle$ diverges $(D = 1/3, U_0 = 66E_R)$

Summary

- Lévy distributions = power-law tail

  $\longrightarrow$ describe large fluctuations

- Strange properties (diverging moments)

  $\longrightarrow$ but observed in nature !

- Direct link between diverging moments and nonergodicity

New statistical physics is needed !