The Full Counting Statistics in Single Electron Transistors

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Outline

- What is the FCS?
- How to calculate the FCS?
- Single Tunneling barrier
- Toward the SET

What is the Full Counting Statistics

- Counting

$$\hat{P}_t(\lambda) = \exp[-S_t(\lambda)] = \sum_n P_t(n)\, e^{in\lambda}$$

$P_{\mathrm{t}}(n)$ probability of n particles are passed in the time t

$S_{\mathrm{t}}(\lambda)$ Cumulant Generating Function (CGF)

Poissonian distribution $P_t(k) = \overline{n}^k e^{-\overline{n}}/k!$, $\quad \overline{n} = \langle I \rangle t$

$$\hat{P}_{\mathrm{t}}(\lambda) = e^{-\overline{n}} \sum_k \frac{\left(\overline{n} e^{i\lambda}\right)^k}{k!} = e^{\overline{n}(\exp[i\lambda]-1)}$$

Binomial distribution $P_t(k)_N = C_N^k\, p^k\, (q)^{N-k}$, $\quad C_N^k = \frac{N!}{(N-k)!k!}$

$$\hat{P}_{\mathrm{t}}(\lambda) = \sum_k C_N^k\, (p\, e^{i\lambda})^k\, (q)^{N-k} = (pe^{i\lambda} + q)^N$$

What is the Full Counting Statistics

- Full $\hat{P}_{\rm t}(\lambda) = e^{-S_{\rm t}(\lambda)} = \sum_n P_t(n)\, e^{in\lambda}$

1. moments
   $$\langle n^k \rangle = \sum_n n^k P_{\rm t}(n) = (-i)^k \frac{\partial^k \hat{P}_{\rm t}(\lambda)}{\partial^k \lambda}\bigg|_{\lambda=0}$$
2. central moments
   $$\mu_k = \langle (n - \overline{n})^k \rangle$$
3. cumulant (*irreducible* correlators)
   $$\kappa_k = (-i)^k \frac{\partial^k \ln[\hat{P}_t(\lambda)]}{\partial^k \lambda}\bigg|_{\lambda=0} = -(-i)^k \frac{\partial^k S_{\rm t}(\lambda)}{\partial^k \lambda}\bigg|_{\lambda=0}$$

Gaussian $S_n \approx i\lambda\overline{n} - \sigma^2\lambda^2/2 \Rightarrow\ \kappa_1 = \overline{n},\ \kappa_2 = \sigma^2, \kappa_{i\geq 3} = 0$

Poissonian $S = \overline{n}(e^{i\lambda} - 1) \Rightarrow\ \kappa_{i\geq 1} = \overline{n}$

Binomial $S_N = N \ln[pe^{i\lambda} + q] \Rightarrow\ \kappa_1 = pN,\ \kappa_2 = pqN,\ \ldots.$

What is the Full Counting Statistics

- Statistics

  X and Y independent$\Rightarrow S_{X+Y} = S_X + S_Y$

  $P_X(\lambda)P_Y(\lambda) = \sum_{x,\,y} P(x)P(y)e^{i(x+y)\lambda} = P_{X+Y}(\lambda)$

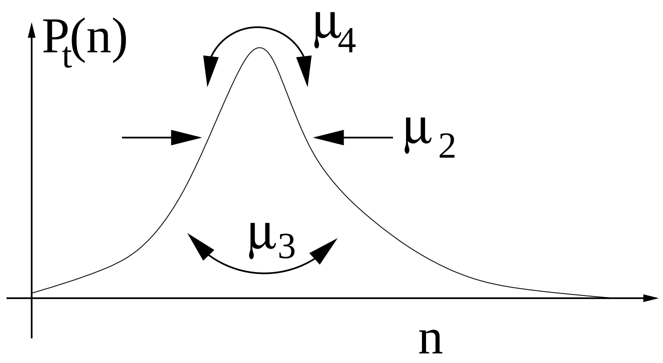
- t measurement time fixed

  $\kappa_1 = \mu_1 = \overline{n}$ *mean*

  $\kappa_2 = \mu_2 = \langle n^2 - \overline{n}^2 \rangle$ *variance*

  $\kappa_3 = \mu_3 = 2\overline{n} - 3\overline{n}\langle n^2 \rangle + \langle n^3 \rangle$ *skewness*

  $\kappa_4 = \mu_4 - 3(\mu_2)^2$ *sharpness* or *kurtosis*



$$\kappa_1 = \overline{n} = \frac{1}{e}\int_0^t dt' \langle I(t') \rangle = \frac{t}{e}\overline{I}$$

$$\kappa_2 = \langle (n - \overline{n})^2 \rangle = \frac{1}{2e^2}\int_0^t dt' \int_0^t dt'' \underbrace{\langle [\delta I(t'), \delta I(t'')]_+ \rangle}_{C_{I^2}(t'-t'')} \underset{t \ggg \tau_c}{=} t\, S_I$$

Belzig (cm/0210125)

Levitov (cm/0210284)

- The skewness

$$\kappa_3 = \frac{1}{6e^2} \int_0^t dt' \int_0^t dt'' \int_0^t dt''' \langle [\delta I(t'), \delta I(t''), \delta I(t''')]_+ \rangle$$
$$\underset{t \ggg \tau_c}{=} t \int_{-\infty}^{+\infty} dt' \int_{-\infty}^{+\infty} dt'' \underbrace{\langle [\delta I(t''), \delta I(t'), \delta I(0)]_+ \rangle}_{C_{I^3}(t', t''-t')}$$

Amplification stage $H_\omega = A$ for $\omega \in (\omega_{min}, \omega_{max})$

1. Noise

$$S_I^{out} = |H_\omega|^2 S_I^{in} \qquad \langle \delta I^2 \rangle = \int d\omega S_I^{out} = A^2(\omega_{max} - \omega_{min}) S_I^{in}$$

2. Skewness

$$C_{I^3}^{out} = \int\int_{-\infty}^{+\infty} \frac{d\omega d\omega'}{2\pi} e^{i(\omega(t+t')+\omega' t')} H_\omega H_{\omega'} H_{-\omega-\omega'} C_{I^3}^{in}(\omega, \omega')$$
$$\langle \delta I^3 \rangle = A^3 \frac{(\omega_{max} - 2\omega_{min})^2}{4\pi^2} C_{I^3}^{in}$$

3. Experiment: Reulet (cm/0302084)

- The sharpness and successive moments

Hard to measure nowadays, experimental problems (?)

How to calculate the FCS

- Spin $1/2$ Galvanometer: Levitov (cm/9607137)

Model vector potential $\vec{A}(r) = \frac{\lambda\Phi_0}{4\pi}\hat{\sigma}_z\vec{\nabla}\theta(f(r) - f_0)$

$$H_i = \frac{1}{c}\int d^3r\vec{j}\cdot\vec{A} = \frac{\lambda\hbar}{2e}\hat{\sigma}_z I$$

λ fictitious coupling constant Nazarov (cm/9908143)

Passive charge detector... after generalized by Kindermann&Nazarov (cm/0107133)

$\rho_s(t) = \text{Tr}_e[e^{-iH_it}\ \hat{\rho}\ e^{iH_it}]$ $\qquad$ $\hat{\rho}_i = \rho_e \otimes \rho_s$

$$\rho_s(t) = \begin{bmatrix} \rho_{\uparrow\uparrow}(0) & \tilde{P}_{-\lambda}\ \rho_{\uparrow\downarrow}(0) \\ \tilde{P}_{+\lambda}\ \rho_{\downarrow\uparrow}(0) & \rho_{\downarrow\downarrow}(0) \end{bmatrix} \qquad \tilde{P}_\lambda = \text{Tr}_e[e^{-iH_{+\lambda}t}\ \rho_e\ e^{iH_{-\lambda}t}]$$

Let‘s introduce the matrix for rotations around z-axis by angle θ

How to calculate the FCS/2

Let‘s suppose we know $P_t(n)$, define the precession angle $\theta = n\lambda$

$$\rho(t) = \sum_n R_\rho(n\lambda) P_t(n) \rho(0)$$

$$R_\rho(\theta) = \begin{bmatrix} \rho_{\uparrow\uparrow} & e^{-i\theta}\rho_{\uparrow\downarrow} \\ e^{i\theta}\rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{bmatrix} \quad \rho_s(t) = \begin{bmatrix} \rho_{\uparrow\uparrow} & \sum_n P_t(n) e^{-in\lambda}\rho_{\uparrow\downarrow} \\ \sum_n P_t(n) e^{in\lambda}\rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{bmatrix}$$

$$\hat{P}_t(\lambda) \equiv \tilde{P}(\lambda) = \langle e^{iH_{-\lambda}t} \; e^{-iH_{+\lambda}t} \rangle$$

Ex. Single channel

$$i\frac{\partial \Psi(x,t)}{\partial t} = \left(\frac{1}{2} \left[-i\frac{\partial}{\partial x} - \frac{\lambda}{2}\delta(x) \right]^2 + U(x) \right) \Psi(x,t)$$

$$\Psi(x) \to e^{i\lambda\theta(x)/2}\Psi(x) \qquad S_\lambda = \begin{bmatrix} A_L e^{i\lambda/2} & B_R \\ B_L & e^{-i\lambda/2} A_R \end{bmatrix}$$

Single Tunneling Barrier

- Leads ($r = R, L$)

$$H^0_{\rm r} = \sum_{r,k} \epsilon_{rk} c^\dagger_{rk} c_{rk}$$

- Tunneling Hamiltonian $H^{\rm T} = \hat{I}_{\rm L\to R} + \hat{I}_{\rm R \to L}$

$$\hat{I}_{\rm L\to R} = (\hat{I}_{\rm R\to L})^\dagger = \sum_{k,k'} \mathbf{t}\, c^\dagger_{Rk} c_{Lk'}$$

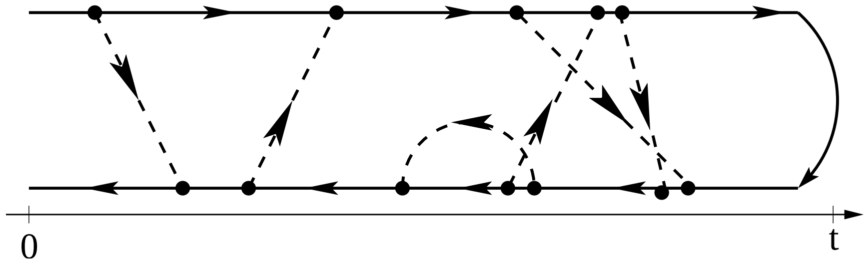
- FCS

$$\hat{P}_t(\lambda) = \mathbf{Tr}[\, T^> \, e^{-i\int_0^t dt' \, H^{\rm I}_\lambda(t')} \, \hat{\rho}_{st} \, T^< \, e^{+i\int_0^t dt' \, H^{\rm I}_{-\lambda}(t')} \,]$$

$$H^{\rm I}_\lambda(t) = H^{\rm I} - i\lambda\,(\hat{I}_{\rm L\to R} - \hat{I}_{\rm R\to L}) \xrightarrow[\Psi \to e^{i\lambda/2}\Psi]{} e^{+i\lambda/2}\,\hat{I}_{\rm L\to R} + e^{-i\lambda/2}\,\hat{I}_{\rm R\to L} = H^{\rm I}_\lambda$$

$$\hat{P}_t(\lambda) = \mathrm{Tr}\,[\, T_{\rm K}\, e^{-i\int_{{\rm K}(0,t)} d\tau \, H^{\rm I}_{\lambda(\tau)}(\tau)} \, \hat{\rho}_{st} \,] \qquad \lambda(t) = \check{\tau}_{\rm K}\,\lambda$$

Single Tunneling Barrier/2



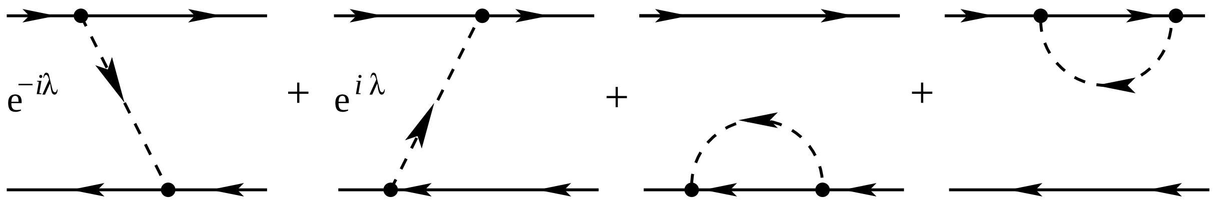
Perturbative expansion

$$\hat{P}_t(\lambda) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_K d\tau_1 \int_K d\tau_2 \ldots \int_K d\tau_n \underbrace{\mathbf{Tr}\left[T_K H^{\mathrm{I}}_{\lambda}(\tau_1)\, H^{\mathrm{I}}_{\lambda}(\tau_2)\, \ldots\, H^{\mathrm{I}}_{\lambda}(\tau_n)\, \hat{\rho}_{st}\right]}_{\mathcal{A}_n}$$

Irreducible self-energy

$$\Sigma_t^{(m)} = \frac{(-i)^m}{m!} \int_K d\tau_1 \int_K d\tau_2 \ldots \int_K d\tau_m \, \mathbf{Tr}\left[T_K H^{\mathrm{I}}_{\lambda}(\tau_1)\, H^{\mathrm{I}}_{\lambda}(\tau_2)\, \ldots\, H^{\mathrm{I}}_{\lambda}(\tau_m)\, \hat{\rho}_{st}\right]_{\mathrm{irr}}$$

In first order in $\Gamma = 2\pi|\mathbf{t}|^2 N(E)$



Single Tunneling Barrier/3

For $n = q_1 m_1 + q_2 m_2 + ... + q_i m_i$

$$\mathcal{A}_n = \sum \prod \mathcal{C}^n_{m_1,q_1;m_2,q_2;....;m_i,q_i} (\Sigma^{(m_1)})^{q_1} (\Sigma^{(m_2)})^{q_2} .....(\Sigma^{(m_i)})^{q_i}$$

where $\mathcal{C}^n_{m_1,q_1;m_2,q_2;....;m_i,q_i} = \frac{n!}{q_1!(m_1!)^{q_1} q_2!(m_2!)^{q_2}...q_i!(m_i!)^{q_i}}$

$$\hat{P}_t(\lambda) = \sum_{q_1,q_2,...} \frac{1}{q_1!} [\Sigma^{(1)}_t]^{q_1} \frac{1}{q_2!} [\Sigma^{(2)}_t]^{q_2} .... = \prod_{i=1}^{\infty} e^{\Sigma^{(i)}_t} = \exp\left[\sum_{i=1}^{\infty} \Sigma^{(i)}_t\right]$$

Using the Wick theorem, from the contractions $\langle c^\dagger_r(t') c_r(t)\rangle$

$$S_t(\lambda) = -\int_0^t dt_1 \int_0^{t_1} dt_2 \; \Sigma(t_1 - t_2)$$

Let‘s calculate the first order contribution

Single Tunneling Barrier/4

$$S_t(\lambda) = (e^{i\lambda} - 1)\; N_{\mathrm{L}\to\mathrm{R}}(t) + (e^{-i\lambda} - 1)\; N_{\mathrm{R}\to\mathrm{L}}(t)$$

Bidirectional Poisson distribution!!!

$$N_{\mathrm{r}\to\bar{\mathrm{r}}}(t) = \int_0^t dt_1 \int_0^t dt_2 \; \langle \; \hat{I}_{\mathrm{r}\to\bar{\mathrm{r}}}(t_1)\; \hat{I}_{\bar{\mathrm{r}}\to\mathrm{r}}(t_2) \; \rangle$$

The charge transferred is $n(t) = \int_0^t dt \hat{I}(t) = -ie \int_0^t dt(\hat{I}_{\mathrm{L}\to\mathrm{R}}(t) - \hat{I}_{\mathrm{R}\to\mathrm{L}}(t))$

Using the Kubo theorem we can write

$$n(t) = \int_0^t dt_1 \int_0^t dt_2 \; \langle \; \left[\hat{I}_{\mathrm{R}\to\mathrm{L}}(t_1),\; \hat{I}_{\mathrm{L}\to\mathrm{R}}(t_2) \right] \; \rangle \; = N_{\mathrm{L}\to\mathrm{R}}(t) - N_{\mathrm{R}\to\mathrm{L}}(t)$$

where obviously $N_{\mathrm{r}\to\bar{\mathrm{r}}}(t) = \Gamma_{\mathrm{r}\to\bar{\mathrm{r}}}\, t$ $\qquad N_{\mathrm{L}\to\mathrm{R}}/N_{\mathrm{R}\to\mathrm{L}} = \Gamma_{1\to 2}/\Gamma_{2\to 1} = \exp[eV/k_{\mathrm{B}}T]$

$$F = \frac{S_I}{2eI} = \frac{\kappa_2}{\kappa_1} = \frac{\Gamma_{1\to 2} + \Gamma_{2\to 1}}{\Gamma_{1\to 2} - \Gamma_{2\to 1}} = \coth\left[\frac{eV}{2k_{\mathrm{B}}T}\right]$$

Levitov-Reznikov(cm/0111057)

Toward the SET

- Electron island (U Coulomb energy)

$$H^0_{\rm dot} = \sum_\sigma \epsilon_\sigma d^\dagger_\sigma d_\sigma + U n_\downarrow n_\uparrow$$

- Double tunneling barrier Hamiltonian + barrier counting fields ($\vec{\lambda} = (\lambda_L, \lambda_R)$)

$$H^{\rm I}_{\vec{\lambda}} = \sum_{r,k,\sigma} \left(t_r \; e^{i\lambda_r(t)/2} \; c^\dagger_{rk\sigma} d_\sigma + h.c. \right)$$

- FCS in SET

$$\hat{P}_t(\vec{\lambda}) = \lim_{T\to-\infty} \mathbf{Tr}[\, T_{\rm K} \; e^{-i\int_{{\rm K}(T,t)} d\tau \; H^{\rm I}_{\vec{\lambda}(\tau)}(\tau)} \; \hat{\rho}_i \,]$$

Now we have the reservoirs (leads) and the dot degrees of freedom!

The initial density matrix $\hat{\rho}$ for the *full system*. With ergodic hypothesis

$$\hat{P}_t(\vec{\lambda}) = \mathbf{Tr}[\, T_{\rm K} \; e^{-i\int_{{\rm K}(0,t)} d\tau \; H^{\rm I}_{\vec{\lambda}(\tau)}(\tau)} \; \rho_\beta \otimes \; \rho_{st} \,]$$

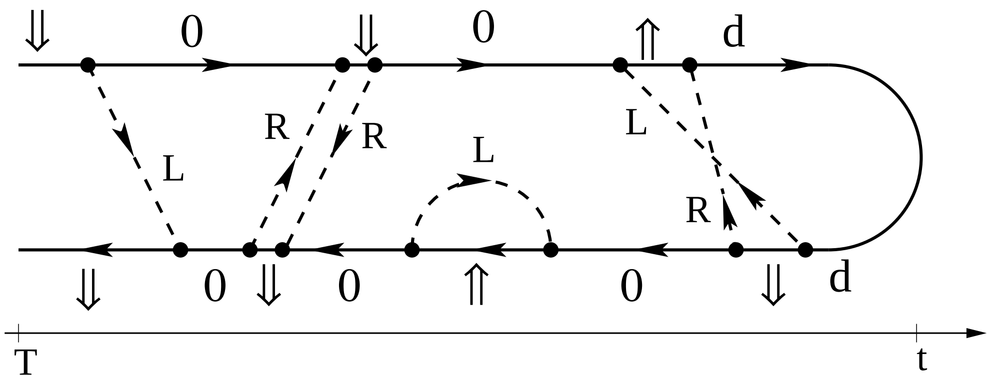
Toward the SET/2

$$\hat{P}_t(\vec{\lambda}) = \langle \mathbf{e^T} | \mathbf{\Pi}_\lambda(t,0)\, | \rho_{st} \rangle$$

Dot density matrix time propagator $\mathbf{\Pi}_\lambda(t,0) = \mathrm{Tr}_{\,leads}[T_K[\, e^{-i\int_{K(0,t)} d\tau\; H^{\mathrm{I}}_\lambda(\tau)}\,]\rho_\beta]$

with $\langle \mathbf{e^T} | = (1,...,1)$ $\qquad |\rho_{st}\rangle$ dot stationary probability

- $\lambda = 0$ Normal diagrammatic rules



Dyson equation for $\mathbf{\Pi}_\lambda(t,0)$, self energy diagrams $\mathbf{W}_\lambda$

$$\mathbf{\Pi}_\lambda(t,0) = \mathbf{1} + \int_0^t dt_2 \int_0^{t_2} dt_1 \mathbf{W}_\lambda(t_2,t_1) \cdot \mathbf{\Pi}_\lambda(t_1,t)$$

Normalization $\quad \hat{P}_t(\mathbf{0}) = \mathbf{e^T} \cdot \mathbf{\Pi_0}(t,0) \cdot \rho_{st} = 1$

Perspective

- full perturbative approach in the tunneling coupling
- compact and unified derivation of all transport properties
- increase the flexibility of the diagrammatic approach
- cotunneling effects

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